Factoring a stabilizer subgroup in a symmetry group

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ABSTRACT

In the study of symmetrical structures, colorings have played an important role, often revealing properties of the structure that may not be immediately apparent. Although a lot of work has been done in relation to colorings there has always been a need for a systematic approach to colorings. In this paper, a symmetry group G which acts on a set X is written as the product of two subgroups A and B of G where $B = Stab_G(x)$, is the stabilizer in G of some element $x \in X$. Transitive colorings of the edges, faces and vertices of the Platonic solids are then obtained using the subgroup A.

1. INTRODUCTION

In the study of a symmetrical structure X and its colorings, it is often assumed that the symmetry group G of X acts regularly on X, i.e., the action of G on X is transitive and the stabilizer of every element of X in G is trivial. This facilitates the study of X and its colorings because a one-to-one correspondence exists between G and X. This correspondence is arrived at by taking a fixed element $x \in X$ and using the assignment $g \mapsto gx$. Because of transitivity, X is the G-orbit $Gx = \{gx : g \in G\}$, of x. If G does not act regularly on X, either G is not transitive on X or G is transitive on X but $Stab_G(x)$ is not trivial. When G is not transitive on X, the standard approach is to work on the distinct G-orbits individually since G acts transitively on each G-orbit. In this paper, we propose an approach in studying X and its colorings when G acts transitively on X but $Stab_G(x)$ is not trivial. The idea is to write G as the product of subgroups A and B where $B = Stab_G(x)$ and $Stab_A(x) = A \cap B$ is minimal, i.e., $A \cap B$ is as close as possible to being the trivial group. Using this idea, a method for determining up to equivalence all transitive colorings of X where the color group acts transitively on X is provided. In particular, all transitive colorings (up to equivalence) of the set X where X is either the set E of edges or the set F of faces or the set V of vertices of a Platonic solid are listed.

The Platonic solids were chosen to illustrate transitive colorings because they are fundamental geometric structures and quite interesting to study. Plato himself showed his great interest by associating the Platonic solids to the elements: the cube to the earth; the tetrahedron to fire; the octahedron to air and the icosahedron to water. Since there are only four elements, Plato associated the dodecahedron to the heavens. [1]

If a coloring is transitive, the associated color group H acts transitively on the set of colors C, that is the H-orbit of each color in C is C. However, there are two possible cases if a coloring is transitive: H is either transitive on the set X or H is not transitive on X. The case where H is not transitive on X may be dealt with using a general framework which was first presented by R. P. Felix in a talk in 1994 and subsequently used by De Las Peñas, Felix and Laigo in [3] to obtain perfect colorings of hyperbolic plane crystallographic patterns. Perfect colorings were used by De las Peñas, Felix and Provido in [4] to determine index 2 subgroups of hyperbolic groups. A continuation of that study, making use of transitive colorings was done by De Las Peñas, Felix and Decena in [2] to determine index 3 and 4 subgroups of hyperbolic symmetry groups. The contribution of this study is to present an easier way of obtaining transitive colorings.

2. PRELIMINARIES

Let X be a symmetrical structure and G its symmetry group. If $C = \{c_1, c_2, \ldots, c_n\}$ is a set of n colors, an onto function $c : X \to C$ is called a **n-coloring of X**. To each $x \in X$ is assigned a color in C. The coloring determines a partition $P = \{P_1, P_2, \ldots, P_n\}$ of X, where P_i is the set of elements of X assigned color c_i. Equivalently, we may think of the coloring as a partition of the set X. For a given coloring of X, we obtain a colored structure $C(X)$. The elements of G which induce a permutation of the colors in the colored structure $C(X)$, form a subgroup H of G which we will refer to as the **color group** determined by the coloring. If $h \in H$ and $c_i, c_j \in C$, we write $hc_i = c_j$ when each $x \in X$ colored c_i is sent to an element colored c_j. This defines an action of H on C which induces a homomorphism f from H to the group of permutations of the set C. The kernel of f is K, a normal subgroup of H and $f(H)$ is isomorphic to the quotient group H/K.

If $S \leq G$ such that $[G : S] = n$ and $\{g_1, g_2, \ldots, g_n\}$ is a complete set of left coset representatives of S in G and $C = \{c_1, c_2, \ldots, c_n\}$ is a set of n colors, the assignment $g_iSx \mapsto c_i$ is a **perfect n-coloring** of X. The set C of

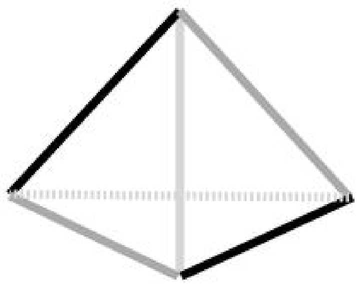

Figure 1: A perfect coloring of the edges of the tetrahedron.

colors form a single orbit under the action of G on C and the coloring is **transitive**, i.e., for any two colors c_i and c_j there is an element $g \in G$ such that $gc_i = c_j$. A **trivial transitive coloring** is a coloring where either all elements of X are assigned a single color or each element of X is assigned a different color. Two colorings of the same symmetrical structure X are said to be **equivalent** if one can be derived from the other by:[5]

(1) an isometry in G,

(2) a bijection from C_1 to C_2 (where C_1 and C_2 are the sets of colors in the first and second colorings, respectively) and

(3) a combination of (1) and (2).

In terms of partitions P and Q of X, the corresponding colorings are said to be equivalent if there exists $g \in G$ such that $Q = gP$.

In this work, the focus is on the situation where G acts transitively on X but $Stab_G(x)$ is not trivial. The idea is to write G as a product AB where $B = Stab_G(x)$ and A is a subgroup of G such that $Stab_A(x) = A \cap B$ is minimal, i.e., as close as possible to being the trivial group. In particular, given the factorization $G = AB$, all transitive colorings of X where the color group H is transitive on X are determined. More specifically, all transitive colorings of X where X is either the set E of edges, or the set F of faces, or the set V of vertices of a Platonic solid are listed.

As an illustration of perfect colorings, consider a regular tetrahedron and take X to be the set E of its edges. The symmetry group G of X is a group of type $\overline{4}3m \cong S_4$. A perfect coloring of E is shown in Figure 1. This is an example of a transitive coloring.

3. METHOD OF OBTAINING COLORINGS

In this section, we will provide a method for obtaining all transitive colorings of X where the color group associated with the coloring acts transitively on X. In the process of obtaining such colorings, the perfect colorings may be extracted since these are just the transitive colorings where $H = G$. The method presented will be based on the theorems that follow.

THEOREM 1. *Let H be a group which acts transitively on a set X and $x \in X$. Let H act transitively on a partition P of X. Then $P = \{hSx : h \in H\}$ for some S such that $Stab_H(x) \leq S \leq H$.*

Proof :

Since H acts transitively on X, there is no loss of generality in taking $x \in X$ and then fixing x. Let $P = \{X_1, X_2, \ldots, X_n\}$ be a partition of X and assume $x \in X_1$. Let $S = Stab_H(X_1) = \{h \in H : hX_1 = X_1\}$. We show $X_1 = Sx$, i.e., X_1 is the S-orbit $\{sx : s \in S\}$ of x. Now $s \in S$ and $x \in X_1$ imply $sx \in sX_1 = X_1$ and hence $Sx \subseteq X_1$. On the other hand, if $y \in X_1$, there is an element $h \in H$ such that $y = hx$ since H acts transitively on X. This implies $X_1 = hX_1$ and so $h \in S$ and $X_1 \subseteq Sx$. Equality of Sx and X_1 follows.

For each $i = 1, \ldots, n$, the transitivity of the action of H on P implies there exists $h_i \in H$ such that $X_i = h_iX_1 = h_iSx$. It follows that $P = \{hSx : h \in H\}$.

The fact that $Stab_H(x) \leq S$ is a consequence of the fact that if $h \in H$ and $hx = x$, then $hX_1 = X_1$ and $h \in S$. ∎

Assume now that the symmetry group G of X acts transitively on X. Pick an element $x \in X$ and let $B = Stab_G(x)$. Write G as a product AB where A is a subgroup of G such that $Stab_A(x) = A \cap B$ is minimal, i.e., $A \cap B$ is as close as possible to the trivial group. Then the transitive colorings of X for which the color group acts transitively on X are given by the partitions $P = \{aSx : a \in A\}$ where $Stab_A(x) \leq S \leq A$ or partitions of the form $\{hS_1x : h \in H\}$ where S_1 is a subgroup of G such that

$$(\bigcup_{i \in I} a_iS)x = S_1x$$

and H is the color group associated with the coloring corresponding to the partition. In effect this result says that it is enough to work inside the subgroup A. This follows from the following two facts:

(1) A is transitive on X since $X = Gx = ABx = Ax$.

(2) If H is the color group and $Stab_H(x) \leq S \leq H$ then an element hsx in hSx where $h \in H$ may be written $hsx = a_hb_ha_sb_sx$ where a_h, $a_s \in A$ and b_h, $b_s \in B$. Since $G = AB = BA$, $b_ha_s = a'b'$ for some $a' \in A$, $b' \in B$. Then $hsx = a_ha'b'b_sx = a_ha'x$ where a_h and $a' \in A$.

We summarize the above discussion in the following theorem.

THEOREM 2. *If G is a group which acts transitively on a set X, $B = Stab_G(x)$ is not trivial and $G = AB$ where $A \leq G$ such that $A \cap B$ is minimal then*

(i) The subgroup A is transitive on X.

(ii) The transitive partitions of X correspond to partitions of the form $\{aSx : a \in A\}$ or $\{hS_1x : h \in H\}$, where $Stab_G(x) \leq S \leq A$, S_1 is a subgroup of G such that

$$(\bigcup_{i \in I} a_i S)x = S_1 x$$

and H is the color group associated with the coloring which corresponds to the partition.

We now formulate the method for determining the transitive colorings of X where the color group associated with the coloring acts transitively on X, G acts transitively on X and $Stab_G(x)$ is finite for $x \in X$.

Fix $x \in X$ and write $G = AB$, where $B = Stab_G(x)$ and A is a subgroup of G such that $Stab_A(x) = A \cap B$ is minimal. There is always a factorization of G in this form, but it may not be unique.

Determine all subgroups S of A such that $Stab_A(x) \subseteq S$. For each subgroup S, form the S-orbit Sx of x. Next, get a complete set Y of left coset representatives of S in A and for each representative $a \in A$, get the image aSx of Sx under a. Assign one color to each aSx, assigning distinct colors to distinct images a_1Sx and a_2Sx. To exhaust all transitive colorings corresponding to partitions $\{aSx : a \in A\}$, let $a' \in A$ and form the orbit $S(a'x)$. If $S(a'x)$ is not congruent to Sx, then get the images of $S(a'x)$ under the elements of Y and assign distinct colors to distinct images of $S(a'x)$. (Two S-orbits are **congruent**, if there is a one-to-one correspondence of their elements and there is an element of G which sends one orbit to the other.)

To determine the transitive colorings corresponding to the partitions of the form $\{hS_1x : h \in H\}$, where S_1 is a subgroup of G such that

$$(\bigcup_{i \in I} a_i S)x = S_1 x,$$

we assign a unique color to S_1x. The images h_1S_1x and h_2S_1x are assigned the same color if and only if $h_1S_1x = h_2S_1x$.

The color group of the transitive colorings derived above contains the subgroup A, so it is transitive on the set X.

To illustrate the method, we determine up to equivalence all transitive colorings of the set E of edges, the set F of faces and the set V of vertices of a Platonic solid where the color group H is transitive on E, F and V.

For the tetrahedron, its symmetry group is $G = \overline{4}3m \cong S_4$, the symmetric group on 4 letters.

(1) If $e \in E$, then $B = Stab_G(e)$ is of type $mm2$.

(2) If $f \in F$, then $B = Stab_G(f)$ is of type $3m$.

(3) If $v \in V$, then $B = Stab_G(v)$ is of type $3m$.

We have the following factorizations for G:

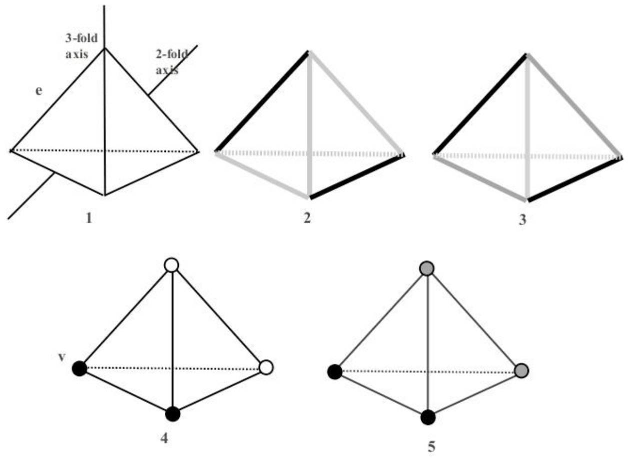

Figure 2: Transitive colorings of the edges and vertices of the tetrahedron.

For (1), $G = AB$, where A is of type $23 \cong A_4$ and is the group of rotations in G.

For (2) and (3), $G = AB$, where A is of type $222 \cong V$ and is the Klein-4 group.

The stabilizer of an edge e in A $Stab_A(e)$ is of type (2), thus the possible subgroups S are $S = 23$, $S = 222$ and $S = 2$. The subgroups $S = 23$ and $S = 2$ will give rise to trivial colorings, so we only consider $S = 222$. If we choose e to be the edge indicated in Figure 2.1, Se is the set of black edges in Figure 2.2. A complete set of left coset representatives of of S in 23 is $\{1, 3, 3^{-1}\}$ thus applying the indicated 3-fold rotation on Se gives us the coloring in Figure 2.3. This coloring of the edges of the tetrahedron is perfect, i.e., the color group is G, which is transitive on the set of edges.

We have similar situations for V and F so let us consider only the derivation of transitive colorings of V. $Stab_A(v) = 1$ for $v \in V$, thus the possible subgroups S of $A = 222$ that we can use are $S = 222$, $S = 2$ and $S = 1$. Again, $S = 222$ and $S = 1$ will give us trivial colorings, so we will only consider $S = 2$. If we choose v to be the vertex indicated in Figure 2.4, the orbit Sv is the set of black vertices. A complete set of left coset representatives of S in $A = 222$ are $\{1, 2\}$. Thus applying the indicated half-turn on Sv gives us the coloring in Figure 2.5. The color group of this particular coloring is the $A = 222$ which is transitive on V.

For the set E of edges, the set V of vertices and the set F of faces of the cube, the symmetry group G is of type $m\overline{3}m$ which is isomorphic to $S_4 \times C_2$.

(1) If $e \in E$, then $B = Stab_G(e)$ is of type $mm2$.

(2) If $v \in V$, then $B = Stab_G(v)$ is of type $3m$.

(3) If $f \in F$, then $B = Stab_G(f)$ is of type $4m$.

We have the following factorizations:

For (1), $G = AB$ where A is of type 23 and is isomorphic to A_4, the alternating group on 4 letters.

For (2), $G = AB$ where A is of type 42 and is isomorphic to D_4, the dihedral group with 8 elements.

For (3), $G = AB$ where A is of type 32 and is isomorphic to D_3, the dihedral group with 6 elements.

For the set E of edges, the set V of vertices and the set F of faces of the dodecahedron, the symmetry group G is of type $\overline{5}3m \cong A_5 \times C_2$.

(1) If $e \in E$, then $B = Stab_G(e)$ is of type $mm2$.

(2) If $v \in V$, then $B = Stab_G(v)$ is of type $3m$.

(3) If $f \in F$, then $B = Stab_G(f)$ is of type $5m$.

We have the following factorizations:

For (1), $G = AB$ where A is of type 532 and is isomorphic to A_5, the alternating group on 5 letters.

For (2), $G = AB$ where A is of type 532 and is isomorphic to A_5.

For (3), $G = AB$ where A is of type 23 and is isomorphic to A_4.

The transitive colorings derived for the cube and the dodecahedron will be presented in tables. The faces, vertices and edges of the cube and the octahedron were numbered as in Figure 3, Figure 4 and Figure 5 respectively, so the tables will contain the subgroup S used, the partition P and whether the corresponding coloring is perfect. By duality of the cube and the octahedron, corresponding colorings of the octahedron may be derived from the colorings of the cube. The duality of these two solids is very useful because the one-to-one correspondence of the vertices and faces of these solids would mean that we only need to color one of the two sets and we can get the coloring of the other set. For the set of edges, there is also a one-to-one correspondence from the set of edges of the cube to the set of edges of the octahedron.

Similarly, because the dodecahedron and the icosahedron are duals, corresponding colorings of the icosahedron may be derived from the colorings of the dodecahedron. The labeling of the faces, vertices and edges of the dodecahedron and icosahedron are given in Figure 6, Figure 7 and Figure 8 respectively.

For the edges of the cube, we get the following non-trivial transitive colorings/partitions of X:

S	Partition P	Color Group
222	{1,2,3,4},{5,7,9,11},{6,8,10,12}	$m\overline{3}m$
3	{1,9,10},{2,6,7},{3,11,12},{4,5,8}	$\overline{4}3m$
3	{1,8,11}, {2,5,12}, {3,6,9},{4,7,10}	432
2	{1,4},{2,3},{5,7},{6,10},{ 8,12},{9,11}	m3
2	{1,3},{2,4},{5,11},{6,12},{7,9},{8,10}	$m\overline{3}m$

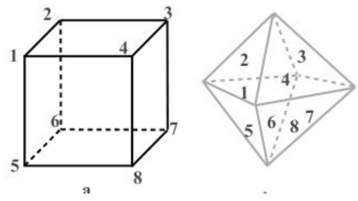

Figure 3: The labeling of the vertices of the cube and the faces of the octahedron where $\{1,2,3,4\}$ are on top and $\{5,6,7,8\}$ are at the bottom.

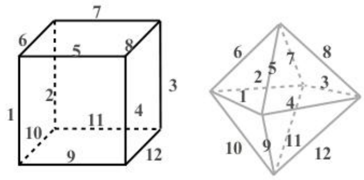

Figure 4: The labeling of the edges of the cube and octahedron where $\{1,4,5,9\}$ are in front, $\{6,8,10,12\}$ are at the sides and $\{2,3,7,11\}$ are at the back.

For the vertices of the cube, we get the following non-trivial transitive colorings/partitions of X:

S	Partition P	Color Group
4	{1,2,3,4},{5,6,7,8}	$4/mmm$
222	{1,3,6,8},{2,4,5,7}	$m\overline{3}m$
222	{1,3,5,7},{2,4,6,8}	$4/mmm$
2	{1,3},{2,4},{5,7},{6,8}	$4/mmm$
2	{1,5},{2,6},{3,7},{4, 8}	$4/mmm$
2	{1,7},{2,8},{3,5},{4, 6}	$m\overline{3}m$

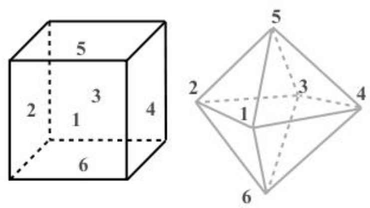

Figure 5: The labeling of the faces of the cube and the vertices of the octahedron where $\{1\}$ is in front, $\{2\}$ is on the left, $\{3\}$ is at the back, $\{4\}$ is on the right, $\{5\}$ is on top and $\{6\}$ is at the bottom.

For the faces of the cube, we get the following non-trivial transitive colorings/partitions of X:

S	Partition P	Color Group
3	{1,5,4}, {2,3,6}	$\overline{3}m$
2	{1,3},{2,4},{5, 6}	$m\overline{3}m$
2	{1,2}, {3,5},{4,6}	32

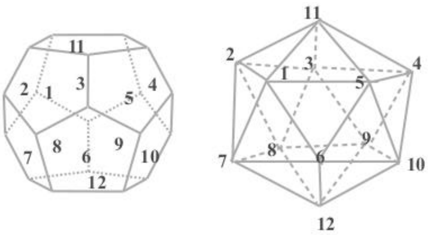

Figure 6: **The labeling of the faces of the dodecahedron and the vertices of the icosahedron where $\{11\}$ is on top, $\{1,2,3,4,5\}$ are on the layer next to the top, $\{6,7,8,9,10\}$ are on the layer before the bottom and $\{12\}$ is at the bottom.**

For the faces of the dodecahedron, we get the following non-trivial transitive colorings/partitions of X:

S	Partition	Color Group
222	{1,9,11,12},{2,5,8,10},{3,4,6,7}	$m3$
3	{1,5,6},{2,3,11},{4,9,10},{7,8,12}	23
3	{1,4,8},{2,6,9},{3,5,12},{7,10,11}	23
2	{1,12},{2,5},{3,10},{4,6},{7,9},{8,11}	23
2	{1,9},{2,10},{3,6},{4,7},{5,8},{11,12}	$\overline{5}3m$
2	{1,2},{3,11},{4,5},{6,12},{7,8},{9,10}	$m3$

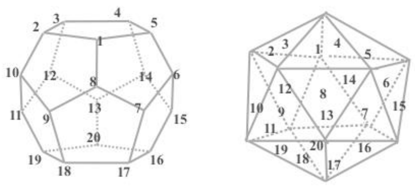

Figure 7: **The labeling of the vertices of the dodecahedron and the faces of the icosahedron where $\{1,2,3,4,5\}$ are on top, $\{6,7,8,9,10,11,12,13,14,15\}$ are in the middle and $\{16,17,18,19,20\}$ are at the bottom.**

For the vertices of the dodecahedron, we get the following non-trivial transitive colorings/partitions of X:

S	Partition	Color Group
23	{1,11,14,17},{2,6,13,18},{3,8,15,19} {4,7,10,20},{5,9,12,16}	532
32	{1,20},{2,16},{3,17},{4,18},{5,19} {6,11},{7,12},{8,13},{9,14},{10,15}	$\overline{5}3m$

For the edges of the dodecahedron, we get the following non-trivial transitive colorings/partitions:

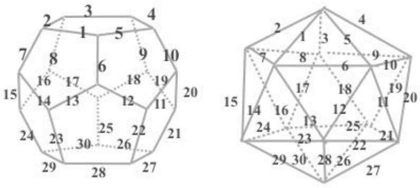

Figure 8: **The labeling of the edges of the dodecahedron and the icosahedron where $\{1,2,3,4,5\}$ are on top, $\{6,7,8,9,10\}$ on the layer next to the top, $\{11,12,13,14,15,16,17,18,19,20\}$ are in the middle, $\{21,22,23,24,25\}$ are at the layer before the bottom and $\{26,27,28,29,30\}$ are at the bottom.**

S	Partition P	Color Group
23	{1,9,11,16,23,26},{2,10,13,18,24,27}, {3,6,15,20,25,28},{4,7,12,17,21,29}, {5,8,14,19,22,30}	$\overline{5}3m$
52	{1,8,12,25,27}, {2,9,14,21,28}, {3,10,16,22,29},{4,6,18,23,30}, {5,7,20,24,26},{11,13,15,17,19}	532
32	{1,19,29}, {2,11,30},{3,13,26}, {4,15,27}, {5,17,28},{6,16,21},{7,18,22}, {8,20,23}, {9,12,24}, {10,14,25}	532
222	{1,26}, {2,27}, {3,28}, {4,29}, {5,30}, {6,25},{7,21},{8,22},{9,23},{10,24}, {11,16},{12,17},{13,18},{14,19},{15,20}	$\overline{5}3m$

4. COMMENTS ON THE LITERATURE

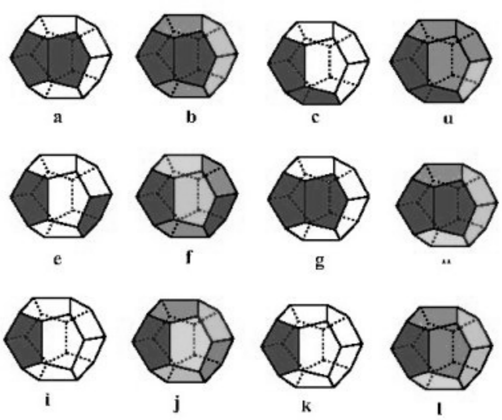

Figure 9: **Transitive colorings of the faces of the pyritohedron.**

An enumeration of all perfect colorings of the face-transitive polyhedra (a class which includes the simple crystal forms) can be found in [8]. For example, the pyritohedron has one 3-coloring ($S = mmm$) and three 6-colorings (two with $S = mm2$ and one with $S = 2/m$). The 12-coloring ($S = m$) is a trivial coloring since the pyritohedron has twelve faces.

Using the method we have presented in the previous section, we factor the symmetry group G of the pyritohedron. Since

G is of type $m3 \cong A_4 \times C_2$ and $B = Stab_G(f)$ is of type m, the group $G = m3$ may be factored as $G = AB$, where A is of type $23 \cong A_4$ and $Stab_A(f) = A \cap B = \{1\}$. We now consider all subgroups S of A. Setting $S = A$ or $\{1\}$ gives trivial colorings so we only consider the normal subgroup 222, the four subgroups of type 3 (two are enough because of conjugacy in A) and the three subgroups of type 2. The S-orbits of the chosen face f is shown in Figures 9a (for $S = 222$); c and e (for $S = 3$); g, i and k (for $S = 3$) and Figures 9b, d, f, h, j, and l show the corresponding colorings.

The perfect colorings of the pyritohedron in the list in [7] may be seen in Figures 9b, h, j and l. Only the 4-coloring is not perfect. To determine the color group H, note that H contains A. Since B is of type m, the coloring will be perfect if and only if m permutes the colors. But m does not. Then $H = A$ which is of type 23.

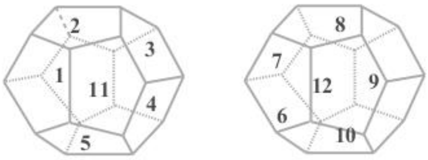

Figure 10: Labels of the faces of the pyritohedron where $\{1, 2, 3, 4, 5, 11\}$ are in front and $\{6, 7, 8, 9, 10, 12\}$ are at the back.

The transitive partitions of the faces of the pyritohedron are given in the following table. The labeling of the faces of the pyritohedron is in Figure 10.

S	Partition	Color Group
222	{1,9,11,12},{2,5,8,10},{3,4,6,7}	$m3$
3	{1,5,6},{2,3,11},{4,9,10},{7,8,12}	23
3	{1,4,8},{2,6,9},{3,5,12},{7,10,11}	23
2	{1,11},{2,8},{3,4},{5,10},{6,7},{9,12}	$m3$
2	{1,9},{2,10},{3,6},{4,7},{5,8},{11,12}	$m3$
2	{1,12},{2,5},{3,7},{4,6},{8,10},{9,11}	$m3$

5. CONCLUSION

In this paper, a method of obtaining transitive colorings was introduced. In this method the transitive colorings of a symmetrical object X were obtained by factoring the symmetry group G of the object as $G = AB$ where $B = Stab_G(x)$ for some $x \in X$ such that $A \cap B$ is minimal. The following are the steps used to obtain the transitive colorings:

1. Fix $x \in X$ and write $G = AB$ where $B = Stab_G(x)$ such that $A \cap B$ is minimal.
2. Determine all subgroups S of A such that $Stab_A(x) \subseteq S$.
3. For each subgroup S do the following:
   (a) Form the S-orbit Sx of x.
   (b) Get a complete set Y of left coset representatives of S in A.
   (c) For each $a \in Y$, get the image aSx of Sx under a
   (d) Assign one color to each aSx, assigning distinct colors to distinct images a_1Sx and a_2Sx.

   Check if there is an element $a' \in A$ such that $S(a'x)$ is not congruent to Sx. If there is, repeat steps (a) to (d) for this S-orbit.

The color group of the transitive coloring is at least the subgroup A.

The factorization of a group G may not be unique. For instance, if we take the set V of vertices of the cube, $Stab_G(v) = 3m$ for $v \in V$. The following are the possible factorizations of $G = m3m$:

$$m3m = (42)(3m)$$

$$m3m = (4/m)(3m)$$

$$m3m = (\bar{4}2m)(3m).$$

The subgroups 42, $4/m$ and $\bar{4}2m$ are the subgroups of $m3m$ of index 6 which are transitive on V.

The same situation is true for the $\{3, 6\}$-tiling of the Euclidean plane (See Figure 11). The symmetry group $G = p6m$ of the tiling may be factored in 2 ways: $p6m = (p2)(3m)$ and $p6m = (p3m1)(3m)$. The groups $p2$ and $p3m1$ are not conjugates in G and for both cases, $A \cap B$ is trivial.

For future work, we plan to implement a computer algorithm based on the algorithm presented in this paper to determine transitive colorings of 2-dimensional tilings of the Euclidean plane. In particular, the algorithmcan be used to determine transitive n-colorings of the (3,6)-tiling in Figure 11.

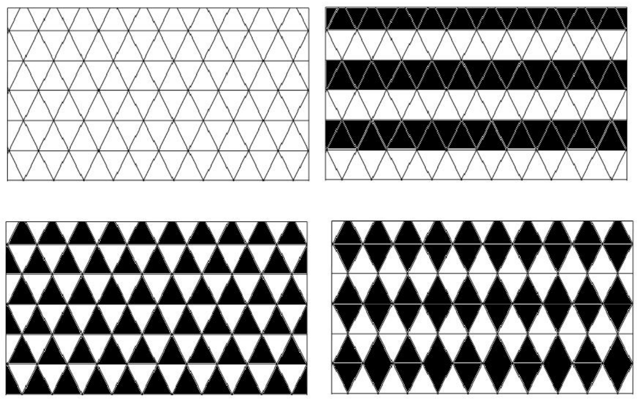

Figure 11: Transitive 2-colorings of the $\{3, 6\}$-tiling with $A = p3m1$.

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