An active set method in total variation based image inpainting

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ABSTRACT

In this paper, we propose a semismooth active set method to image inpainting. The method exploits primal and dual features of a proposed regularized L^2 total variation model. The model and the solution method is based on the work of Hintermueller and Stadler in [8]. Numerical results show that the method is fast and efficient in inpainting sufficiently thin domains. We show that the method works comparably well in relation to the recent split Bregman method.

1. INTRODUCTION

One of the aspects of image processing is inpainting. This is the process of filling-in removed, damaged, or unwanted regions in images. Image inpainting is synonymous with image interpolation wherein continuously defined data is constructed on a region in such a way that the region blends well with surrounding features. In [1], Bertalmio et al first applied inpainting to digital images by using high order PDE models. Since then, numerous approaches to inpainting have been developed: variational techniques , wavelet-based methods, elastica model, isotropic diffusion, etc. See, e.g., [3, 2, 6, 12]. In [4], Cai, Osher, and Shen presented a split Bregman approach to solving a variation model for inpainting, and in [10], a median filter was introduced.

In [5], Chan and Shen developed the following total variation model that inpaints non-texture type images:

$$\min_{u\in BV(E\cup D)} \alpha \int_{E\cup D} |\nabla u|\ dx + \frac{1}{2}\int_E (u-u_0)^2\ dx \qquad (1)$$

where the observed image is denoted by u_0, E is any fixed closed domain outside D, and $|\cdot|$ denotes the Euclidean norm. The image domain $E\cup D$ is taken to be a square. The model is closely related to the Rudin, Osher, and Fatemi (ROF) model for image denoising. Since the TV model is nondifferentiable, Chan and Shen introduced a global smoothing parameter to the TV term and a steady solution is obtained using a low pass filter and a Gauss-Jordan iteration scheme.

A variational model for image reconstruction on a rectangular image domain Ω with Lipschitz continuous boundary $\partial\Omega$ is

$$\min_{u\in BV(\Omega)} \frac{\mu}{2}\int_\Omega |\nabla u|_2^2 + \frac{1}{2}\int_\Omega |Ku-d|^2 + \alpha\int_\Omega |\nabla u|_2.$$

In [8], Hintermueller and Stadler regularized the above model by local smoothing on the TV term, i.e., by replacing $\int_\Omega |\nabla u|$ with

$$\mathcal{F}_\gamma(\nabla u) = \begin{cases} \dfrac{1}{2\gamma}|\nabla u|_2^2 & \text{if } |\nabla u|_2 < \gamma \\ |\nabla u|_2 - \dfrac{\gamma}{2} & \text{if } |\nabla u|_2 \geq \gamma \end{cases}$$

where $\gamma > 0$. The resulting model is

$$\min_{u\in BV(\Omega)} \frac{\mu}{2}\int_\Omega |\nabla u|_2^2 + \frac{1}{2}\int_\Omega |Ku-d|^2 + \alpha\int_\Omega \mathcal{F}_\gamma(\nabla u). \quad (2)$$

Further, they developed a semismooth Newton-type method that solves the resulting regularized version of the TV model using an active set strategy. The method was shown to converge superlinearly. In this paper, we used a modified regularized variational model (2) that is amenable to image inpainting. The primary change is in the restriction of the fidelity term $\int \frac{1}{2}|u-d|^2$ to the non-inpainting domain E. We develop an active-set approach to solve the resulting model and we show that the results of the method compares well with those of the split Bregman method recently presented by Goldstein and Osher [11].

2. MODEL

We discretized the image domain $E\cup D$ to an $n\times n$ pixel-grid. For ease in computation, we concatenate the columns of the image matrix u to an image vector $v\in\mathbb{R}^N, N=n^2$. The discrete total variation of v is formulated as

$$\begin{aligned} TV(v) &= \sum_{l=1}^N |[\nabla v]_l| \\ &= \sum_{l=1}^N \sqrt{(\nabla_x v)_l^2 + (\nabla_y v)_l^2} \end{aligned} \qquad (3)$$

where $[\nabla v]_l = [(\nabla_x v)_l, (\nabla_y v)_{l+n}]^T$. The gradient components ∇_x and ∇_y are approximated using forward differences. The discrete total variation image inpainting model for (1) is

$$\min_{v\in\mathbb{R}^N} TV(v) + f(v) = \alpha \sum_{l=1}^{N} |[\nabla v]_l| + \frac{1}{2}|v - v^0|^2_{\{\tilde{E}\}} \quad (4)$$

where we define $|x|_{\{W\}} = \sqrt{\sum_{i\in W} x_i^2}$ for an index set W.

Let v^0 be the observed image in stacked form. We propose the following regularized variation model for inpainting:

$$\min_{v\in\mathbb{R}^N} \left\{ \alpha\mathcal{F}_\gamma(\nabla v) + \frac{1}{2}|v - v^0|^2_{\{\tilde{E}\}} + \frac{\mu}{2}\sum_{l=1}^{N}|[\nabla v]_l|^2 \right\} \quad (\mathcal{P}_\gamma)$$

where

$$\mathcal{F}_\gamma(\nabla v) = \begin{cases} \frac{\lambda}{2\gamma}\sum_{l=1}^{N}|[\nabla v]_l|^2 & \text{if } |[\nabla v]_l| < \gamma \\ \lambda\sum_{l=1}^{N}\left(|[\nabla v]_l| - \frac{\gamma}{2}\right) & \text{if } |[\nabla v]_l| \geq \gamma \end{cases}$$

for $l = 1, 2, \ldots, N$. Clearly, model $(\mathcal{P}_\gamma)$ is convex and is guaranteed a unique solution. When the inpainting region D is empty, $(\mathcal{P}_\gamma)$ reverts to the working model in [8].

The active set method that we implement exploits the primal-dual features of $(\mathcal{P}_\gamma)$, whose Fenchel pre-dual is

$$\sup_{p\in\mathbb{R}^{2N}, ||[p]_l||\leq\lambda} \left\{ -\frac{1}{2}|||\text{div } p + v^0|||^2_{\{\tilde{E}\}} + \frac{1}{2}||v^0||^2_{\{\tilde{E}\}} - \frac{1}{2}|||\text{div } p|||^2_{\{\tilde{D}\}} \right\} \quad (\mathcal{D}_\gamma)$$

where

$$|||x|||^2_{\{X\}} = \begin{cases} \langle (I_N - \mu\Delta)^{-1}x, x\rangle & \text{if } X = \tilde{E}, \\ \langle (-\mu\Delta)^{-1}x, x\rangle & \text{if } X = \tilde{D}. \end{cases} \quad (5)$$

$\triangle$ is the discrete Laplacian, and $\text{div} = -\nabla^\top$ is the discretized divergence (cf. [9]). Note that in (5), the restriction to the specified pixel set indicates that only terms corresponding to indices in the pixel set are evaluated.

3. OPTIMALITY CONDITIONS

The solutions to the primal $(\mathcal{P}_\gamma)$ and dual $(\mathcal{D}_\gamma)$ problems, given by $\bar{v}_\gamma$ and $\bar{p}_\gamma$ respectively, satisfy the following optimality conditions:

$$-\mu\Delta\bar{v}_\gamma + \bar{v}_\gamma - \text{div } \bar{p}_\gamma = v^0 \quad \text{on } \tilde{E} \quad (6)$$

$$-\mu\Delta\bar{v}_\gamma = \text{div } \bar{p}_\gamma \text{ on } \tilde{D} \quad (7)$$

$$\begin{cases} \gamma[\bar{p}_\gamma]_l - \lambda[\nabla\bar{v}_\gamma]_l = 0 & \text{if } ||[\bar{p}_\gamma]_l|| < \lambda \\ ||[\nabla\bar{v}_\gamma]_l||\ [\bar{p}_\gamma]_l = \lambda[\nabla\bar{v}_\gamma]_l & \text{if } ||[\bar{p}_\gamma]_l|| = \lambda \end{cases} \quad \text{on } \tilde{E}\cup\tilde{D} \quad (8)$$

for $l = 1, \ldots, N$.

Let $\kappa \in \mathbb{R}^N$ with $\kappa_i = 1$ if pixel-index $i \in E$; 0 otherwise. We combine equations (6) and (7) as

$$-\mu\Delta\bar{v}_\gamma - \text{div } \bar{p}_\gamma + \kappa_{\tilde{E}}(\bar{v}_\gamma - v^0) = 0 \quad (9)$$

where $\kappa_{\tilde{E}} = D(\kappa)$, the $N\times N$ diagonal matrix.

The optimal conditions in (8) can also be combined as:

$$\max(\gamma, ||[\nabla\bar{v}_\gamma]_l||)\,[\bar{p}_\gamma]_l - \lambda\nabla[\bar{v}_\gamma]_l = 0 \quad (10)$$

for every $l = 1, 2, \ldots, N$. In the next section, we present a Newton-type solution method based on the optimality conditions presented here.

4. A SEMISMOOTH METHOD

Using equations (9) and (10), we determine a Newton method that mirrors the active set approach in [8] for image denoising. Results in generalized differentiability and semismoothness (cf. [7]) allow the use of a Newton step to (6),(7), and (10) at the k-th approximations v^k and p^k :

$$\begin{pmatrix} -\mu\Delta + \kappa_{\tilde{E}} & -\text{div} \\ G\nabla & D(m^k) \end{pmatrix}\begin{pmatrix} \delta_v \\ \delta_p \end{pmatrix} = \begin{pmatrix} \mu\Delta v^k + \text{div } p^k + \kappa_{\tilde{E}}(-v^k + v^0) \\ \lambda\nabla v^k - D(m^k)p^k \end{pmatrix} \quad (11)$$

where

$$G = -\lambda I_{2N} + \chi_{\mathcal{A}_{k+1}} D(p^k)J(\nabla v^k)$$

$$m^k = \max(\gamma I_{2N}, \eta(\nabla v^k)) \in \mathbb{R}^{2N}$$

with the mapping $\eta : \mathbb{R}^{2N} \mapsto \mathbb{R}^{2N}$ given by

$$(\eta(v))_i = ||v_i|| \text{ with } v \in \mathbb{R}^{2N}, i = 1, \ldots, 2N$$

Now, the active set indicator $\chi_{\mathcal{A}_{k+1}} = D(t^k)$ which is a $2N\times 2N$ diagonal matrix with

$$t_i^k := \begin{cases} 1 & \text{if } (\eta(\nabla v^k))_i \geq \gamma \\ 0 & \text{if } (\eta(\nabla v^k))_i < \gamma \end{cases}$$

determines whether a component is part of the active set $\mathcal{A}_{k+1}$, by setting $t_i = 1$, or not. The matrix J is the Jacobian of η, that is

$$J(\nabla v) = (D(\eta(\nabla v)))^{-1}\begin{pmatrix} D(\nabla_x v) & D(\nabla_y v) \\ D(\nabla_x v) & D(\nabla_y v) \end{pmatrix}$$

With all components of $m^k > 0$, this means that the diagonal matrix $D(m^k)$ is invertible. We obtain δ_p and δ_v as

$$\delta_p = \lambda D^{-1}(m^k)\nabla v^k - p^k - D^{-1}(m^k)G\nabla\delta_v \quad (12)$$

and

$$H_k\delta_v = f_k \quad (13)$$

with

$$H_k = -\mu\Delta + \kappa_{\tilde{E}} + \text{ div } D^{-1}(m^k)G\nabla$$

$$f_k = \mu\Delta v^k + \text{ div } \lambda D^{-1}(m^k)\nabla v^k + \kappa_{\tilde{E}}\left(-v^k + v^0\right).$$

Whenever H_k is not positive definite, we use the modifications in [8] to get the positive definite matrix H_k^+.

We propose the following active set method for inpainting:

Algorithm: Active Set Method

1. Set $k = 0$ and initialize $(v^0, p^0) \in \mathbb{R}^N \times \mathbb{R}^{2N}$.

2. Determine the members of the active set by solving $\chi_{\mathcal{A}_{k+1}} \in \mathbb{R}^{2N \times 2N}$.

3. Compute H_k^+ if p^k is not feasible for all $i = 1, \ldots, N$. Otherwise set $H_k^+ = H_k$.

4. Solve for δ_v in $H_k^+ \delta_v = f_k$ and compute δ_p.

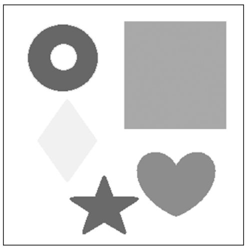
5. Update $v^{k+1} = v^k + \delta_v$ and $p^{k+1} = p^k + \delta_p$.

6. Stop, or set $k := k + 1$ and go to step 2.

We note that the proposed method for inpainting is analogous to that in [8] for denoising. Numerical implementations of the algorithm are presented next.

5. NUMERICS

The algorithm is implemented in MATLAB R2011a on a machine with a speed of 3.40 GHz and with 8 GB of RAM. Our test images are square grayscale images which are nearly noise-free and blur-free degraded only by thin lines and text which are the inpainting domains. The goal of inpainting is to reconstruct the inpainting domain by using the image information surrounding these domains.



(a) Original Image

Figure 1: Image Example 1

There is no ideal value for γ; however, the smaller γ is, the better the observed inpainting and restoration of edges. With this, we set $\gamma = 10^{-4}$. The value of α is set to 0.01. In assessing the performance of our method, we shall use the Split Bregman method which splits problem (4) into subproblems in order to find its minimizer. This is an efficient method capable of remarkable results in the least amount of time. Refer to [11] for the discussion of the method and [13] for its implementation.

Our first image sample is a 300×300 image (figure 1(a)). The image masked with thin lines (about 2 to 4 pixels wide) is figure 2(a). The mask is user-defined and is created using an image editing software. The benchmark method obtained the result shown in figure 2(b) after 23 iterations in 0.87 seconds with a residual of 7.471. The result obtained using the active set method is shown in figure 2(c). This result is obtained at 2 iterations, with a time of 2.9 seconds. The method converged in 14 iterations with a residual of 4.699. Convergence is determined once the norm of the vector composed of the left-hand-sides of optimality conditions (9) and (10) has sufficiently decreased from its initial value. The residual of the proposed method is lower than that of the benchmark method. On closer inspection of the reconstructed images, one can see that the active set method did a better job of inpainting the lines. In particular, the criss-crossed lines in the square are still evident in the split Bregman result whereas these were completely inpainted in the active set result. There also appears a lightening of the colors in the split Bregman than in the active set method.

The second masked image has text to be inpainted (figure 3(a)).

Using the Split Bregman method, the inpainted image 3(b) is acquired at 20 iteration for 0.7 seconds with a residue of 3.860. The reconstruction by the active set method is given in figure 3(c) is obtained in 2 iterations and 2.8 seconds. The method converged after 12 iterations in 21.3 seconds incurring a residue of 2.503. Again, upon inspection, the active set method did a better inpainting job, specifically in the donut shape.

The second image example is a 300×300 grayscale image (figure 4(a)). Thin lines constitute the inpainting domain. The reconstruction using the active set method effectively removed the lines in 9 iterations with a time of 14.3 seconds and residue of 5.527. The split bregman method took 26 iterations in 0.89 seconds to obtain the figure 4(c) with a residue of 5.704. The reconstruction of the active set method fared well with the split Bregman. However, the band effect of the variational model is more evident in this reconstruction than with the other method.

6. CONCLUSION

We presented a variation model for image inpainting and a semismooth primal-dual active set method to solve it. Our numerical experiments show that the method is very effective in providing good reconstructions. Also, the method is at par if not better with known methods such as the split bregman method, though, the proposed method do have some drawbacks concerning with its runtime. The algorithm is applicable for filling in small domains in non-texture based images.

7. ACKNOWLEDGEMENT

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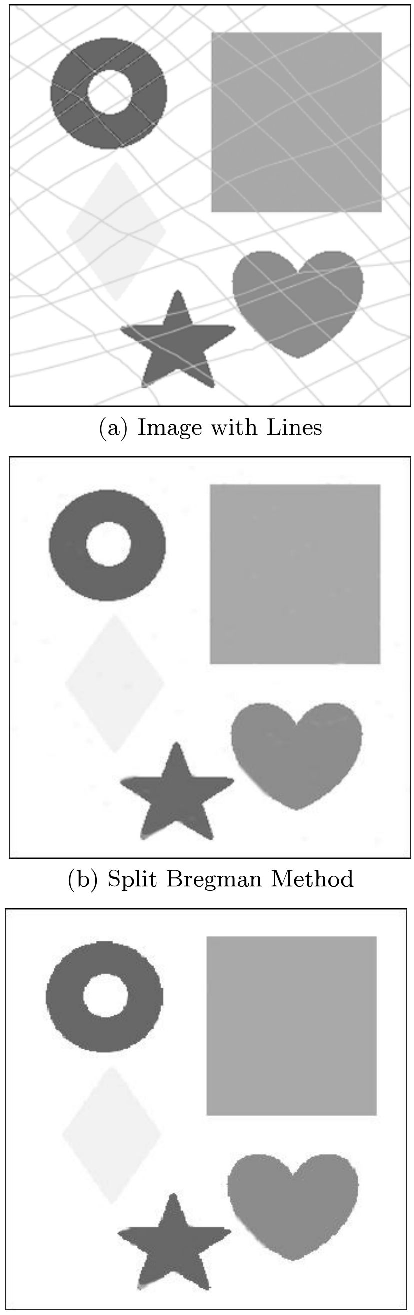


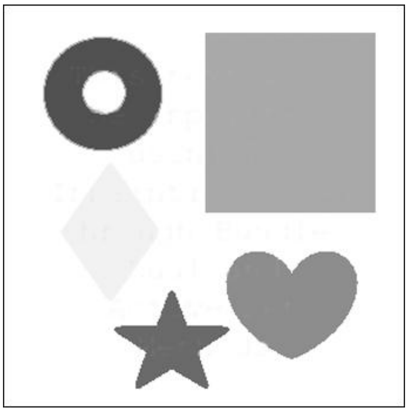
(a) Image with Lines

(b) Split Bregman Method

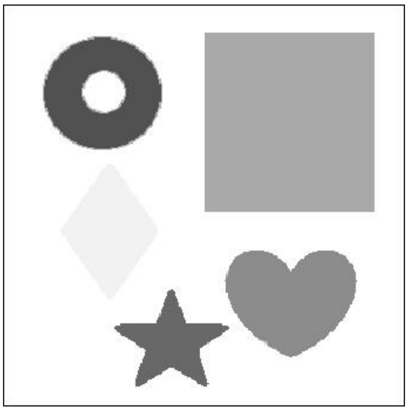
(c) Active Set Method

Figure 2: Inpainting lines

(a) Image with Text



(b) Split Bregman Method



(c) Active Set Method

Figure 3: Inpainting text

(a) Original Image

(b) Degraded Image

(c) Split Bregman Method

(d) Active Set Method

Figure 4: Image Example 3