The Lucci Cube: A New Graph and Some of Its Structural and Enumerative Properties

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ABSTRACT

Fibonacci cubes and Lucas cubes have been investigated extensively in several journal articles [1, 4, 9, 10], with the Fibonacci cube being introduced to be an alternative to the well-known hypercube as an interconnection topology for parallel systems [4].

In this paper, a new graph called the Lucci cube, created by modifying the definitions of the Fibonacci and Lucas cubes such that no two 0's are in the first and last bits of binary strings simultaneously, is studied. Properties of the Lucci cube such as structural decomposition, order, size, radius, center, diameter, and maximum and minimum degrees are determined. In addition, we discuss the hamiltonicity and independence numbers of this graph.

Keywords

Fibonacci cube, Lucas cube

1. INTRODUCTION

This research paper will discuss definitions of Fibonacci string, Lucas string, Fibonacci cube, Lucas cube, Lucci string, and Lucci cube. The main objective of this paper is to generate a formula for the order and size of a Lucci cube, to determine structural properties such as radius, center, diameter, and maximum and minimum degrees, and to discuss its hamiltonicity, and vertex- and edge-independence.

Basic graph-theoretic terminologies used in this paper are adopted from [2].

Definition 1.

1. A graph G is a finite nonempty set of objects called vertices together with a (possibly empty) set of unordered pairs of distinct vertices of G called edges. The vertex set of G is denoted by $V(G)$, while the edge set is denoted by $E(G)$. The cardinalities $|V(G)|$ and $|E(G)|$ are called the order and the size of G, respectively.
2. If $e = \{u, v\}$ is an edge of a graph G, then u and v are said to be adjacent. The edge is also denoted uv.

We will also make use of the well-known Fibonacci and Lucas numbers. The nth Fibonacci number F_n and the nth Lucas number L_n are the numbers described by the following recurrence relations:

$$F_n = F_{n-2} + F_{n-1} \text{ for } n > 2, \text{ where } F_1 = F_2 = 1$$
$$L_n = L_{n-2} + L_{n-1} \text{ for } n > 2, \text{ where } L_1 = 1 \text{ and } L_2 = 3$$

The first few Fibonacci and Lucas numbers then are as follows:

$$(F_n)_{n>0} = (1, 1, 2, 3, 5, 8, 13, 21, \ldots)$$
$$(L_n)_{n>0} = (1, 3, 4, 7, 11, 18, 29, 47, \ldots)$$

2. FIBONACCI AND LUCAS CUBES

We present the Fibonacci and Lucas cubes as defined in [10] and [9] together with some results on these graphs.

Definition 2. [10]

1. A Fibonacci string of length n is a binary string $b_1 b_2 \ldots b_n$ containing no two consecutive 1's.

2. The nth Fibonacci cube Γ_n is the graph defined as follows:
   (a) The vertex set of Γ_n is the set of all Fibonacci strings of length n.
   (b) Two vertices are adjacent in Γ_n if they differ in exactly one bit.

In Figure 1 are the first five Fibonacci cubes with their corresponding orders $|V(\Gamma_n)|$.

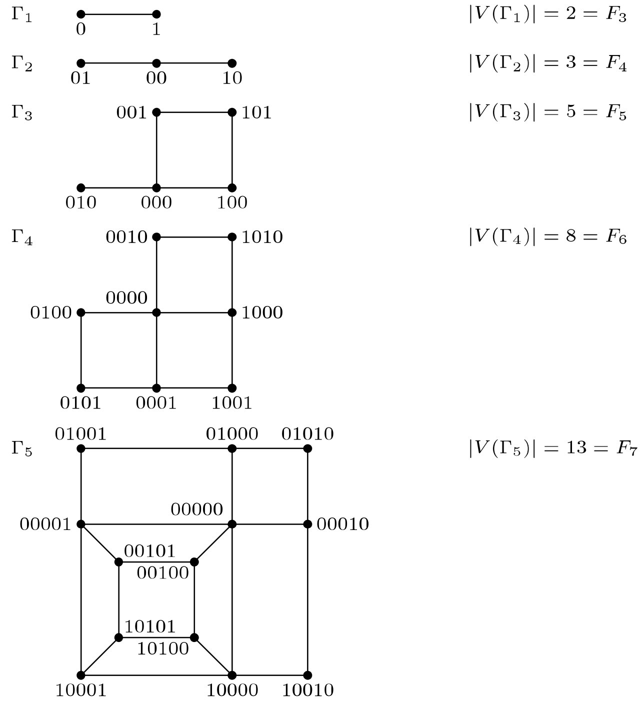

Figure 1: The first five Fibonacci cubes

The Fibonacci cubes are so named because of their orders, given in the following statement.

THEOREM 1. [4] *For every positive integer n, $|V(\Gamma_n)| = F_{n+2}$.*

To find the number of edges of Γ_n, the following property of the Fibonacci cube is of much help.

THEOREM 2. [4] *The Fibonacci cube Γ_n can be decomposed into two vertex-disjoint subgraphs A and B, with $A \cong \Gamma_{n-1}$ and $B \cong \Gamma_{n-2}$, where for every $v \in V(B)$, there is exactly one $u \in V(A)$ such that $uv \in E(\Gamma_n)$.*

Thus, we have the following theorem, giving a recurrence relation and an explicit formula for the size f_n of Γ_n.

THEOREM 3. [4] *The size f_n of Γ_n is given by:*

1. $f_n = f_{n-1} + f_{n-2} + F_n$, *for* $n \geq 3$
2. $f_n = \dfrac{nF_{n+1} + 2(n+1)F_n}{5}$, *for* $n \geq 2$

The Lucas cubes have the same adjacency relation as the Fibonacci cubes, but have a stricter condition for membership in the vertex set.

Definition 3. [9]

1. A Lucas string of length n is a binary string $b_1b_2\ldots b_n$ containing no two consecutive 1's such that the first and the last bits are not simultaneously 1.
2. The nth Lucas cube Λ_n is the graph defined as follows:
   (a) The vertex set of Λ_n is the set of all Lucas strings of length n.
   (b) Two vertices are adjacent in Λ_n if they differ in exactly one bit.

In Figure 2 are the first five Lucas cubes with their corresponding orders $|V(\Lambda_n)|$. Similarly, the Lucas cubes are so named because of their orders.

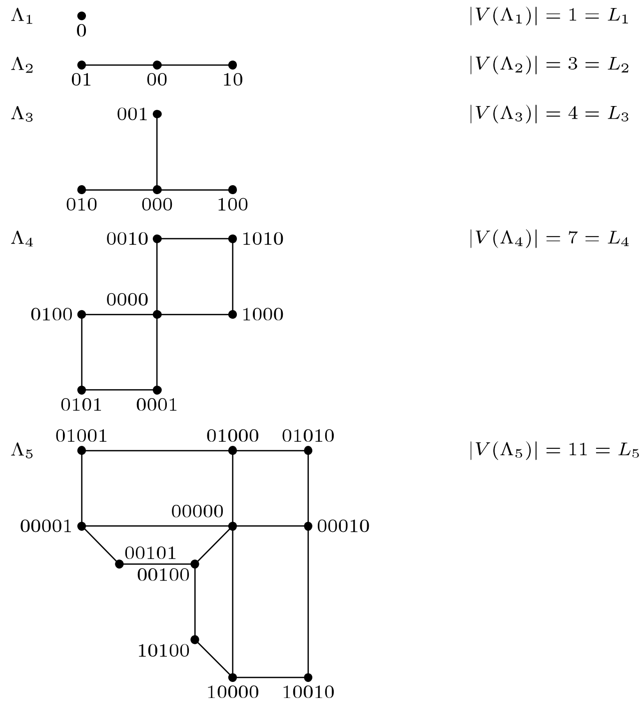

Figure 2: The first five Lucas cubes

THEOREM 4. [9] *For every positive integer n, $|V(\Lambda_n)| = L_n$.*

THEOREM 5. [9] *The nth Lucas cube Λ_n can be decomposed into two vertex-disjoint subgraphs A and B, with $A \cong \Gamma_{n-1}$ and $B \cong \Gamma_{n-3}$, where for every $v \in V(B)$, there is exactly one $u \in V(A)$ such that $uv \in E(\Lambda_n)$.*

THEOREM 6. [9] *The size l_n of Λ_n is given by:*

1. $l_n = f_{n-1} + f_{n-3} + F_{n-1}$, *for* $n \geq 4$
2. $l_n = nF_{n-1}$, *for* $n \geq 3$

3. THE LUCCI CUBE

We define a new graph by modifying the condition for membership in the vertex set. This time we disallow the first and last bits to be 0 simultaneously.

Definition 4.

1. A Lucci string of length n is a binary string $b_1 b_2 \ldots b_n$ containing no two consecutive 1's such that the first and the last bits are not simultaneously 0.
2. The nth Lucci cube ψ_n is the graph defined as follows:
   (a) The vertex set of ψ_n is the set of all Lucci strings of length n.
   (b) Two vertices are adjacent in ψ_n if they differ in exactly one bit.

Shown in Figure 3 are the first 7 Lucci cubes together with their corresponding orders $|V(\psi_n)|$. From these first few examples, it appears that $|V(\psi_n)| = F_{n+1}$. It will be proved that this is true indeed for any positive integer n.

3.1 Decomposition, Order, Size

First, we prove a decomposition theorem for the Lucci cube analogous to Theorems 2 and 5. We adopt the following notation from [10]. Let α and β be binary strings. The concatenation of α and β is denoted by $\alpha\beta$. If S is a set of binary strings, let $\alpha S \beta = \{\alpha s \beta \mid s \in S\}$. Then, letting $C_n = V(\Gamma_n)$, $10C_n0$ is the vertex set of a graph isomorphic to Γ_n, and the vertices are formed by appending 10 to the start and 0 to the end of each vertex of Γ_n. The corresponding graph will be denoted $10\Gamma_n0$.

THEOREM 7. *If $n \geq 5$, then ψ_n can be decomposed into three vertex-disjoint subgraphs A, B and C, with $A \cong B \cong \Gamma_{n-3}$ and $C \cong \Gamma_{n-4}$ such that for every $w \in V(C)$, there is exactly one $u \in V(A)$ such that $uw \in E(\psi_n)$ and exactly one $v \in V(B)$ such that $vw \in E(\psi_n)$.*

PROOF. A Lucci string starts with either 0 or 1.

Case 1. Suppose a Lucci string starts with 0. Then, it must end with a 1, which must be preceded by a 0. Thus, such a string has the form $0b01$, where b may be any binary string of length $n-3$ that does not have consecutive 1's; that is, b may be any element of C_{n-3}. Therefore, the subgraph induced by these vertices is isomorphic to Γ_{n-3}.

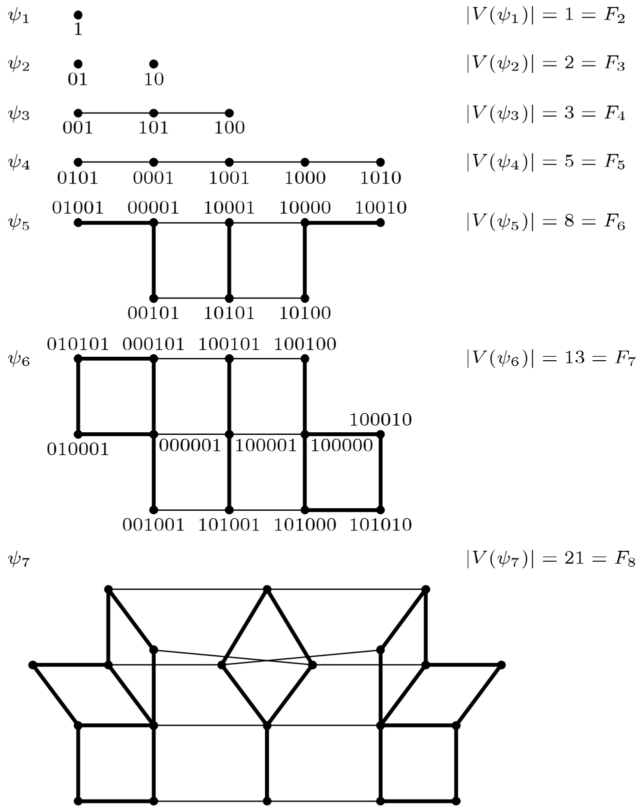

Figure 3: The first seven Lucci cubes

Case 2. Suppose a Lucci string starts with 1. Then, the second bit must be 0, so the string has the form $10b$, where b may be any element of C_{n-2}. Now, b may end with either 0 or 1.

Case 2.1 If the Lucci string ends with 0, then it would have the form $10b0$, where $b \in C_{n-3}$. The subgraph induced by the vertices in this subcase then is isomorphic to Γ_{n-3}.

Case 2.2 If the Lucci string ends with 1, then the second to the last bit must be 0, yielding $10b01$, where $b \in C_{n-4}$. The subgraph induced by the vertices in this subcase then is isomorphic to Γ_{n-4}.

We take $A = 0\Gamma_{n-3}01$, $B = 10\Gamma_{n-3}0$ and $C = 10\Gamma_{n-4}01$. Note that A and B are not linked by any edge since a vertex in A will differ in the first and the last bits with a vertex in B.

Let $w \in V(C)$. Then, $w = 10b01$ for some $b \in C_{n-4}$, which is adjacent to $u = 00b01 \in V(A)$, and it is clear that any other vertex in A differs in at least two bits from w. Similarly, the only vertex in B adjacent to w is $v = 10b00$. □

The decomposition theorem thus gives us an algorithm to obtain ψ_n for $n \geq 5$. For illustrations, see the 5th, 6th and 7th Lucci cubes in Figure 3. The thickened portions correspond to A, B and C, and the slim edges are those that link A and B to C.

The theorem also connotes an attractive geometric symmetry in the Lucci cube. To demonstrate this symmetry, place $C = 10\Gamma_{n-4}01$ in the center, between $A = 0\Gamma_{n-3}01$ and $B = 10\Gamma_{n-3}0$. Given a binary string b, let $\bar{b}$ denote the string obtained by reversing its bits. For every $0b01$ in $V(A)$, there corresponds a vertex $10\bar{b}0$ in $V(B)$, and vice-versa. Meanwhile, any edge from A to C must link vertices of the form $00b01$ and $10b01$. To this edge corresponds one joining B to C, namely the edge formed by $10\bar{b}00$ and $10\bar{b}01$. These correspondences between $V(A)$ and $V(B)$, and between the edges from A to C and the edges from B to C establishes geometric symmetry in ψ_n for $n \geq 5$. Again, this is illustrated by the 5th, 6th and 7th Lucci cubes in Figure 3. Though the theorem holds only for $n \geq 5$, we see from the same figure that the Lucci cubes with $1 \leq n \leq 4$ also possess symmetry. Thus, geometric symmetry in ψ_n holds for $n \geq 1$.

Furthermore, because there are edges that link A and B to C, the graph is connected when $n \geq 5$. The graph is also connected for $n = 3$ and 4.

COROLLARY 1. *The nth Lucci cube ψ_n is connected for $n \geq 3$.*

The decomposition theorem also implies that topologies that can be embedded in Fibonacci cubes can also be embedded in the Lucci cube, thus the Lucci cube can be considered as an interconnection topology for multiprocessor systems.

Another importance of the theorem is its use in obtaining explicit formulas for the order and the size of the Lucci cube.

THEOREM 8.

1. $|V(\psi_n)| = F_{n+1}$
2. $|E(\psi_n)| = 2f_{n-3} + f_{n-4} + 2F_{n-2}$ *for* $n \geq 5$
3. $|E(\psi_n)| = \dfrac{(n+4)F_n + 2(n-5)F_{n-1}}{5}$ *for* $n \geq 2$

PROOF.

By the decomposition of the Lucci cube, if $n \geq 5$, then

$$\begin{aligned}|V(\psi_n)| &= 2|V(\Gamma_{n-3})| + |V(\Gamma_{n-4})| = 2F_{n-1} + F_{n-2} \\ &= F_n + F_{n-1} = F_{n+1}\end{aligned}$$

Moreover, this could be verified by substitution for $1 \leq n \leq 4$.

Furthermore, if $n \geq 5$, then

$$\begin{aligned}|E(\psi_n)| &= 2|E(\Gamma_{n-3})| + |E(\Gamma_{n-4})| + 2|V(\Gamma_{n-4})| \\ &= 2f_{n-3} + f_{n-4} + 2F_{n-2}\end{aligned}$$

We now establish the explicit formula for the size of ψ_n. Suppose $n \geq 5$. Then, using the recurrence relation for F_n,

$$\begin{aligned}|E(\psi_n)| &= 2\frac{(n-3)F_{n-2} + 2(n-2)F_{n-3}}{5} \\ &\quad + \frac{(n-4)F_{n-3} + 2(n-3)F_{n-4}}{5} + 2F_{n-2} \\ &= \frac{(2n+4)F_{n-2} + (5n-12)F_{n-3} + (2n-6)F_{n-4}}{5} \\ &= \frac{(4n-2)F_{n-2} + (3n-6)F_{n-3}}{5} \\ &= \frac{(3n-6)F_{n-1} + (n+4)F_{n-2}}{5} \\ &= \frac{(n+4)F_n + 2(n-5)F_{n-1}}{5}\end{aligned}$$

By substitution, the result can be verified easily for $n = 2$, 3 and 4. □

We conclude this portion with a remark worth noting. Notice that if $n \geq 2$, Γ_n and ψ_{n+1} have the same order: F_{n+2}. However,

$$\begin{aligned}|E(\Gamma_n)| - |E(\psi_{n+1})| &= \frac{nF_{n+1} + 2(n+1)F_n}{5} \\ &\quad - \frac{(n+5)F_{n+1} + 2(n-4)F_n}{5} \\ &= \frac{-5F_{n+1} + 10F_n}{5} \\ &= 2F_n - F_{n+1} \\ &= F_n - F_{n-1} > 0\end{aligned}$$

Thus, while the topologies use up the same number of vertices, the $(n+1)$th Lucci cube uses less connections.

3.2 Radius, Center, Diameter

We determine other structural properties of ψ_n which are important in viewing graphs as interconnection topologies. We also define the following terms.

Definition 5. Let G be a connected graph.

1. The eccentricity $e_G(v)$ of a vertex v of G is the maximum distance of any vertex of G from v.
2. The radius rad(G) of G is the minimum eccentricity among the vertices of G.
3. The center $Z(G)$ of G is the set of all vertices in G whose eccentricity is equal to the radius.
4. The diameter diam(G) of G is the greatest possible distance between any two vertices of G.

Note that the adjacency rule of the Fibonacci, Lucas and Lucci cubes implies that the distance between any two vertices in these graphs is the number of bits in which they differ.

We will also be adopting the following notation for binary strings from [1]. If b is a binary string of length k, the string $b^{\frac{n}{k}}$ is formed by concatenating $\left[\!\left[\frac{n}{k}\right]\!\right]$ copies of b and an additional copy of the first r bits of b, where $n \equiv r \mod k$, if $0 < r < k$. In other words, $b^{\frac{n}{k}}$ is the string formed by writing the string b repeatedly until n bits are written. For example, $(101)^{\frac{10}{3}} = 1011011011$.

THEOREM 9. *For any $n \geq 3$,*

1. $\text{rad}(\psi_n) = \left[\!\left[\frac{n}{2}\right]\!\right]$
2. $Z(\psi_n) = \{1(0)^{n-2}1\}$.

PROOF. The results on the radius and center can be verified for $n = 3$ or 4. Let $n \geq 5$. Consider $c = 1(0)^{n-2}1$. A vertex of maximal distance from c has the form $10v_1$ or $v_2 01$, where v_1 is a vertex of maximal distance from $(0)^{n-4}01$ in Γ_{n-2} and v_2 is a vertex of maximal distance from $1(0)^{n-3}$ in Γ_{n-2}. To construct v_1, replace the 1 in $(0)^{n-4}01$ by 0 and then replace the remaining $(n-3)$-string of zeros by a Fibonacci string of maximal distance from $(0)^{n-3}$ in Γ_{n-3}. The vertex v_2 can be obtained by a similar argument. Since $(0)^k \in Z(\Gamma_k)$ and $\text{rad}(\Gamma_k) = \left[\!\left[\frac{k+1}{2}\right]\!\right]$ [10], we obtain the distance $d(c, 10v_1) = d(c, v_2 01) = 1 + \left[\!\left[\frac{n-2}{2}\right]\!\right] = \left[\!\left[\frac{n}{2}\right]\!\right]$. Since $10v_1$ and $v_2 01$ are of maximal distance from c, $e_{\psi_n}(c) = \left[\!\left[\frac{n}{2}\right]\!\right]$.

Let $v \in V(\psi_n)$, $v \neq c$. In order to conclude that $Z(\psi_n) = \{c\}$, we show that $e_{\psi_n}(v) > e_{\psi_n}(c)$ by constructing a vertex v^* that is at a distance greater than $\left[\!\left[\frac{n}{2}\right]\!\right]$.

By Theorem 7, v is a vertex in $0C_{n-3}01$, $10C_{n-3}0$ or $10C_{n-4}01$. Suppose $v \in 0C_{n-3}01$. Either v is in $00C_{n-4}01$ or in $010C_{n-5}01$. (For the case when $n = 5$, the only vertex of the second form is 01001, and $e_{\psi_n}(01001) = 4 > \left[\!\left[\frac{5}{2}\right]\!\right]$). Suppose $v \in 00C_{n-4}01$. Let $v = 00s01$, and k be the number of 1's in s. We construct v^* by replacing the first 0 in v with 1, the last 1 with 0, all the 1's in s with 0, and the string of $(n-4)-k+1$ 0's formed by the 0's in the substring $s0$ by a Fibonacci string of maximal distance from this string as a vertex of $\Gamma_{(n-4)-k+1}$. Then, $d(v, v^*) = 1 + 1 + k + \left[\!\left[\frac{n-k-2}{2}\right]\!\right] > \left[\!\left[\frac{n}{2}\right]\!\right]$. Therefore, $e_{\psi_n}(v) > e_{\psi_n}(c)$.

Suppose $v \in 010C_{n-5}01$. Let $v = 010s01$ and k be the number of 1's in s. We construct v^*, this time by replacing the 01 at the beginning by 10, the last 0 by 1, the k 1's in s by 0, and the $(n-5)-k+2$ string of 0's in the substring $0s0$ by a Fibonacci string of maximal distance from it as a vertex of $\Gamma_{(n-5)-k+2}$. This v^* is also at a distance greater than $\left[\!\left[\frac{n}{2}\right]\!\right]$ from v. This proves that $e_{\psi_n}(v) > e_{\psi_n}(c)$, in this case. We repeat the above arguments for the case when $v \in 10C_{n-3}0$, leaving the case when $v \in 10C_{n-4}01$.

If $v \in 10C_{n-4}01 \setminus \{c\}$, then it has the form $10s01$, where $s \in C_{n-4}$ and has at least one 1. Let k be the number of 1's in s. Either $k \geq 2$ or $k = 1$. If $k \geq 2$, we construct v^* by replacing the last 1 in v with 0, the k 1's in s by 0, and the remaining $(n-4)-k+1$ string of 0's in $s0$ by a Fibonacci string of maximal distance from it. Then, $d(v, v^*) = 1 + k + \left[\!\left[\frac{n-k-2}{2}\right]\!\right] \geq \left[\!\left[\frac{n}{2}\right]\!\right]$. Since $k \geq 2$, equality of these expressions is impossible.

If $k = 1$ and n is odd, v^* is constructed in the same fashion, and $d(v, v^*) = 1 + 1 + \left[\!\left[\frac{n-1-2}{2}\right]\!\right] = 2 + \frac{n-3}{2} > \left[\!\left[\frac{n}{2}\right]\!\right]$. This leaves the case when $k = 1$ and n is even.

The sole 1 in s is either in an odd or an even bit position, counting from the leftmost bit. Without loss of generality, assume it is in an odd position, and consider $v^* = (01)^{\frac{n}{2}}$. Then, $d(v, v^*) = \frac{n}{2} + 1 > \left[\!\left[\frac{n}{2}\right]\!\right]$. Therefore, in all cases, there exists a v^* such that $d(v, v^*) > e_{\psi_n}(c)$. This proves that $e_{\psi_n}(v) > e_{\psi_n}(c)$.

Hence, the eccentricity of any vertex of ψ_n different from c is greater than that of c. Therefore, $\text{rad}(\psi_n) = \left[\!\left[\frac{n}{2}\right]\!\right]$ and $Z(\psi_n) = \{c\}$. □

THEOREM 10. *For any $n \geq 3$,*

1. $\text{diam}(\psi_n) = \begin{cases} n & , \text{ if } n \text{ is even} \\ n-1 & , \text{ if } n \text{ is odd} \end{cases}$
2. *the number of pairs of vertices which are at a distance equal to the diameter is 1 if n is even and $n-2$ if n is odd.*

PROOF. We know that ψ_n is connected for $n \geq 3$. If n is even, then the vertices $(10)^{\frac{n}{2}}$ and $(01)^{\frac{n}{2}}$ differ in all n positions, hence $\text{diam}(\psi_n) = n$. Moreover, this is the only pair of vertices that have distance n. If n is odd, the only binary strings that differ in all n positions are $(10)^{\frac{n}{2}}$ and $(01)^{\frac{n}{2}}$. However, the second string has zeros in both the first and last bits, and so is not a vertex in ψ_n. Thus, $\text{diam}(\psi_n) \leq n-1$. But there are vertices, in particular $001(01)^{\frac{n-3}{2}}$ and $100(10)^{\frac{n-3}{2}}$, which differ in exactly $n-1$ bits. Hence, $\text{diam}(\psi_n) = n-1$ for n odd.

Furthermore, if n is odd, then a distance of $n-1$ between two vertices can be achieved only if the vertices each have at least $\frac{n-1}{2}$ 1's. But, the vertex $v = (10)^{\frac{n}{2}}$ cannot have a vertex of distance $n-1$ from it since any vertex has the same first two bits, or the same last two bits as v. Thus, the maximum number of 1's in an n-Lucci string that has a vertex that is of distance $n-1$ from it, where n is odd, is $\frac{n-1}{2}$.

There are two types of these vertices: those of the form $(10)^{\ell}0(10)^{\frac{n-1}{2}-\ell}$ and those of the form $(01)^{\frac{n-1}{2}-\ell}0(01)^{\ell}$ where $1 \leq \ell \leq \frac{n-1}{2}$. Now, $v_1 = 100(10)^{\frac{n-3}{2}}$ is of the first form, and the only vertex of distance $n-1$ from this is $v_2 = 001(01)^{\frac{n-3}{2}}$, which is of the second form. Conversely, the only vertex of distance $n-1$ from v_2 is v_1.

For each $2 \leq \ell \leq \frac{n-1}{2}$, there are two vertices of distance $n-1$ from $(10)^{\ell}0(10)^{\frac{n-1}{2}-\ell} = (10)^{\ell-1}(100)(10)^{\frac{n-1}{2}-\ell}$, namely $(01)^{\ell-1}(010)(01)^{\frac{n-1}{2}-\ell}$ and $(01)^{\ell-1}(001)(01)^{\frac{n-1}{2}-\ell}$, which are both of the second form. This gives an additional $n-3$ pairs of vertices of distance $n-1$. Arguing similarly, any vertex in the second case besides v_2 has exactly 2 vertices of distance $n-1$ from it, and both are of the first form. Hence, there is no new pair of vertices formed, resulting in a total of $n-2$ pairs of vertices having distance $n-1$. □

3.3 Maximum and minimum degrees

Another useful property of a graph G is its maximum degree $\Delta(G)$ and minimum degree $\delta(G)$, where the degree of a vertex is the number of vertices adjacent to it. It is clear that $\Delta(\psi_n) = n-2$ for $n \geq 2$, where the vertices of maximum degree are $1(0)^{n-1}$ and $(0)^{n-1}1$. Meanwhile, it is known that $\delta(\Gamma_k) = \left[\!\left[\frac{k+2}{3}\right]\!\right]$ [4]. Now, by Theorem 7 and a further decomposition of Γ_{n-3}, ψ_n can be decomposed into $10\Gamma_{n-5}010$, $10\Gamma_{n-4}00$, $10\Gamma_{n-4}01$, $00\Gamma_{n-4}01$ and $010\Gamma_{n-5}01$, where every vertex in each subgraph is adjacent to at least one vertex outside the subgraph it belongs to. Hence, $\delta(\psi_n) \geq 1 + \delta(\Gamma_{n-5}) = \left[\!\left[\frac{n}{3}\right]\!\right]$. But there exists a vertex, namely $10(010)^{\frac{n-5}{3}}010$, that is of degree $\left[\!\left[\frac{n}{3}\right]\!\right]$ in ψ_n. Therefore, $\delta(\psi_n) = \left[\!\left[\frac{n}{3}\right]\!\right]$ for $n \geq 6$, and this can be verified for $2 \leq n \leq 5$. Thus, we have the following statement.

THEOREM 11. *Let $n \geq 2$.*

1. $\Delta(\psi_n) = n-2$
2. $\delta(\psi_n) = \left[\!\left[\frac{n}{3}\right]\!\right]$

3.4 Enumerative Properties

Because of the adjacency rule for the Lucci cube, if a vertex has an odd number of 1's, then any vertex adjacent to it must have an even number of 1's, and conversely. This leads us to distinguish between vertices with an even number of 1's and those with an odd number of 1's.

Definition 6. Let $n \geq 1$.

1. $\overline{E}_n$ = set of vertices of ψ_n with an even number of 1's
2. $\overline{O}_n$ = set of vertices of ψ_n with an odd number of 1's
3. $\overline{e}_n = |\overline{E}_n|$
4. $\overline{o}_n = |\overline{O}_n|$
5. $\overline{h}_n = \overline{e}_n - \overline{o}_n$

Note that $\overline{E}_n \cup \overline{O}_n = V(\psi_n)$ and $\overline{E}_n \cap \overline{O}_n = \varnothing$, and each of the two sets of vertices are independent, that is, no two vertices in each set are adjacent. In this case, we say that the sets $\overline{E}_n$ and $\overline{O}_n$ form a bipartition $(\overline{E}_n, \overline{O}_n)$ of $V(\psi_n)$.

Define $\overline{e}_0$ to be 1 and $\overline{o}_0$ to be 0. Thus, $\overline{h}_0 = 1$. We give the values of $\overline{e}_n$, $\overline{o}_n$ and $\overline{h}_n$ for the first few nonnegative integers.

n	0	1	2	3	4	5	6	7
$\overline{e}_n$	1	0	0	1	3	5	7	10
$\overline{o}_n$	0	1	2	2	2	3	6	11
$\overline{h}_n$	1	-1	-2	-1	1	2	1	-1

Table 1: The values of $\overline{e}_n$, $\overline{o}_n$ and $\overline{h}_n$ for small values of n.

The analogues for Fibonacci cubes of the sets and quantities defined above are denoted by E_n, O_n, e_n, o_n and h_n, with $e_0 = 1$, $o_0 = 0$. We shall use the following results.

LEMMA 1. [10] *For $n \geq 0$, $h_{n+2} = h_{n+1} - h_n$, $h_{n+3} = -h_n$ and $h_{n+6} = h_n$.*

Analogous properties also hold for the Lucci cube. To this end, we first prove the following statement.

LEMMA 2. *For $n \geq 0$, $\overline{e}_{n+3} = o_{n+1} + o_n$ and $\overline{o}_{n+3} = e_{n+1} + e_n$.*

PROOF. A Lucci string having length $n+3$ and an even number of 1's must be either of the form $10b$, where $b \in C_{n+1}$ and has an odd number of 1's, or of the form $0b01$, where $b \in C_n$ and has an odd number of 1's. This shows that $\overline{e}_{n+3} = o_{n+1} + o_n$ for $n \geq 1$, and this can be easily verified for $n = 0$. The result for $\overline{o}_n$ is derived from a similar argument. □

Using these facts, we can derive the results in the following theorem, including the closed form of the ordinary generating function $\overline{H}(x) := \sum_{n=0}^{\infty} \overline{h}_n x^n$ for the sequence $(h_n)_{n\geq 0}$.

THEOREM 12. *The sequence $(\overline{h}_n)_{n\geq 0}$ satisfies the following properties:*

1. *$\overline{h}_{n+3} = -h_{n+1} - h_n$, for $n \geq 0$*
2. *$\overline{h}_{n+3} = -\overline{h}_n$, for $n \geq 0$*
3. *$\overline{h}_{n+6} = \overline{h}_n$, for $n \geq 0$, and the repeated values are 1, -1, -2, -1, 1 and 2.*
4. *The ordinary generating function for $(\overline{h}_n)_{n\geq 0}$ is $\overline{H}(x) = \dfrac{1-2x}{1-x+x^2}$.*

PROOF.

1. By Lemma 2,
$$\begin{aligned}\overline{h}_{n+3} &= \overline{e}_{n+3} - \overline{o}_{n+3} \\ &= o_{n+1} + o_n - e_{n+1} - e_n \\ &= -h_{n+1} - h_n\end{aligned}$$
2. By the previous statement and Lemma 1, $\overline{h}_{n+3} = -h_{n+1} - h_n = h_{n-2} + h_{n-3} = -\overline{h}_n$.
3. By the previous statement, $\overline{h}_{n+6} = -\overline{h}_{n+3} = \overline{h}_n$. Thus, $(\overline{h}_n)_{n\geq 0}$ has period 6 and the repeated values are as given in Table 1.
4. The ordinary generating function $\overline{H}(x)$ for $\overline{h}_n$ is $\overline{H}(x) := \sum_{n=0}^{\infty} \overline{h}_n x^n$. Before proceeding, note that by Lemmas 1 and 2,
$$\begin{aligned}\overline{h}_n &= -h_{n-2} - h_{n-3} \\ &= -h_{n-1} - h_{n-2} - h_{n-3} + h_{n-1} \\ &= \overline{h}_{n+1} - h_{n-3} - h_{n-4} \\ &= -\overline{h}_{n-2} + \overline{h}_{n-1}\end{aligned}$$

Thus, $\overline{h}_n - \overline{h}_{n-1} + \overline{h}_{n-2} = 0$, giving us

$$\begin{aligned}(1-x+x^2)\overline{H}(x) &= \overline{h}_0 + \overline{h}_1 x + \sum_{n=2}^{\infty} \overline{h}_n x^n - \overline{h}_0 x \\ &\quad - \sum_{n=1}^{\infty} \overline{h}_n x^{n+1} + \sum_{n=0}^{\infty} \overline{h}_n x^{n+2} \\ &= \overline{h}_0 + (\overline{h}_1 - \overline{h}_0)x \\ &\quad + \sum_{n=2}^{\infty} (\overline{h}_n - \overline{h}_{n-1} + \overline{h}_{n-2})x^n \\ &= 1 + (-1-1)x = 1-2x\end{aligned}$$

□

3.5 Hamiltonicity and Independence Numbers

Recall that a path in a connected graph G is said to be Hamiltonian if it passes through all vertices of G (exactly once), and a cycle in G is said to be Hamiltonian if it contains all vertices of G. A graph is said to be Hamiltonian if it has a Hamiltonian cycle.

The fact that $(\overline{E}_n, \overline{O}_n)$ is a bipartition of $V(\psi_n)$ is useful in deriving the following result.

THEOREM 13.

1. *ψ_n is never Hamiltonian.*
2. *ψ_n has a Hamiltonian path if and only if $n \not\equiv 2 \mod 3$.*

PROOF. Since $(\overline{E}_n, \overline{O}_n)$ is a bipartition of $V(\psi_n)$, a Hamiltonian cycle must alternate between vertices in either set and end where it begins. If the vertices are to be exhausted in this process, the two sets must have the same number of elements. Since from Theorem 12, $|\overline{e}_n - \overline{o}_n| = |\overline{h}_n| = 1$ or 2 only, ψ_n cannot be Hamiltonian. This proves the first statement.

It is easy to see that ψ_1 (trivial), ψ_3 and ψ_4 have a Hamiltonian path, while ψ_2 does not. Now, assume $n \geq 5$.

Suppose $n \equiv 2 \mod 3$. First, from the bipartition of $V(\psi_n)$, any path in ψ_n must alternate between vertices in $\overline{E}_n$ and $\overline{O}_n$. Since $|\overline{e}_n - \overline{o}_n| = |\overline{h}_n| = 2$ for $n \equiv 2 \mod 3$, there cannot exist a path passing through all the vertices of ψ_n.

Suppose $n \not\equiv 2 \mod 3$. We will be using the fact that for any m, Γ_m has a Hamiltonian path beginning at $0(100)^{\frac{m-1}{3}}$ and ending at $(100)^{\frac{m}{3}}$ [1]. Clearly, the path tracing this in reverse is also a Hamiltonian path.

Assume first that $n \equiv 0 \mod 3$. Arguing that the last bit of a Lucci string is either 0 or 1, one could see that any vertex has the form $10b0$ where $b \in C_{n-3}$, or $b01$ where $b \in C_{n-2}$. Thus, ψ_n can be decomposed into $10\Gamma_{n-3}0$ and $\Gamma_{n-2}01$. A Hamiltonian path in $\Gamma_{n-2}01$ beginning at $0(100)^{\frac{n-3}{3}}01$ and ending at $(100)^{\frac{n-2}{3}}01$ exists. The last vertex is adjacent to $(100)^{\frac{n-2}{3}}00$, a vertex in $10\Gamma_{n-3}0$. This vertex be expressed as $100(100)^{\frac{n-4}{3}}0$ because $n \equiv 0 \mod 3$. Now, a Hamiltonian path in $10\Gamma_{n-3}0$ beginning at $100(100)^{\frac{n-4}{3}}0$ and ending at $10(100)^{\frac{n-3}{3}}0$ exists. Stringing these two paths together makes a Hamiltonian path in ψ_n.

Assume $n \equiv 1 \mod 3$. By Theorem 7, ψ_n can be decomposed into $10\Gamma_{n-3}0$, $10\Gamma_{n-4}01$ and $0\Gamma_{n-3}01$. A Hamiltonian path exists in $10\Gamma_{n-3}0$, beginning at $10(100)^{\frac{n-3}{3}}0$ and ending at $100(100)^{\frac{n-4}{3}}0$. Since $n \equiv 1 \mod 3$, $n-5 \equiv 2 \mod 3$, and so $100(100)^{\frac{n-4}{3}}0$ can also be expressed as $100(100)^{\frac{n-5}{3}}00$. This last vertex is adjacent to $100(100)^{\frac{n-5}{3}}01$, a vertex in $10\Gamma_{n-4}01$. A Hamiltonian path exists in $10\Gamma_{n-4}01$, beginning at $100(100)^{\frac{n-5}{3}}01$ and ending at $10(100)^{\frac{n-4}{3}}01$. This last vertex is adjacent to $00(100)^{\frac{n-4}{3}}01$, a vertex in $0\Gamma_{n-3}01$. A Hamiltonian path in $0\Gamma_{n-3}01$ that begins at $00(100)^{\frac{n-4}{3}}01$ and ends at $0(100)^{\frac{n-3}{3}}01$ exists. Stringing together these three paths, a Hamiltonian path in ψ_n is constructed. □

Recall that a trail in a nontrivial connected graph G using all the edges of G exactly once is called an Eulerian trail. If the first and last vertices of the trail are the same, then it is called an Eulerian circuit, and the graph is said to be Eulerian. Two classical results in graph theory are that a graph is Eulerian if and only if all of its vertices have even degree, and that a non-Eulerian graph has an Eulerian trail if and only if it has exactly two vertices of odd degree.

THEOREM 14. *ψ_n is not Eulerian, and has an Eulerian trail only when $n = 3$ or 4.*

PROOF. The result is easily verified for $n = 1$, 2, 3 and 4. Suppose $n \geq 5$. It suffices to show that ψ_n has at least three vertices of odd degree, because then the graph would not have an Eulerian trail, much less an Eulerian circuit.

Case 1. Suppose $n = 2k+1$, $k \geq 2$. We claim that $u = 1(0)^{2k}$, $v = (0)^{2k}1$ and $w = 1(0)^{2k-1}1$ each have degree $2k-1$. The vertices adjacent to u are formed by changing any of the last $2k-1$ 0's to 1. The first two bits cannot be changed. Thus, the degree of u is $2k-1$. Similarly, for v, only the first $2k-1$ 0's may be changed. For w, only the first and the last 0 cannot be changed so the degree of w is $2k-1$ indeed.

Case 2. Suppose $n = 4k$, $k \geq 2$. We claim that $u = (10)^{2k}$, $v = (01)^{2k}$, and $w = (10)^{2k-3}(100)^2$ each have degree $2k-1$. For u, only the last $2k-1$ 1's may be changed. For v, only the first $2k-1$ 1's may be changed. For w, only the last $2k-2$ 1's and the last 0 may be changed. That proves the claim.

Case 3. Suppose $n = 4k+2$, $k \geq 1$. We claim that $u = (10)^{2k}00$, $v = 00(01)^{2k}$ and $w = (10)^{2k}01$ each have degree $2k+1$. For u, only the last $2k-1$ 1's and the last two 0's may be changed. For v, only the first $2k-1$ 1's and the first two 0's may be changed. For w, only the $2k+1$ 1's may be changed. This completes the proof. □

REMARK. Though it was not explicitly stated in papers on Γ_n and Λ_n, among these graphs only Λ_4 is Eulerian. Among the rest, only Γ_1, Γ_2, Γ_3, Γ_4 and Λ_2 have an Eulerian trail.

We conclude this paper with the independence numbers of ψ_n. Recall that $X \subseteq V(G)$ is said to be independent if no two vertices in X are adjacent, and $Y \subseteq E(G)$ is said to be independent if no two edges in Y have a common vertex. The vertex independence number $\alpha(G)$ of G is the largest cardinality of an independent set of vertices of G, while the edge independence number $\alpha'(G)$ of G is the largest cardinality of an independent set of edges of G.

THEOREM 15.

1. $\alpha'(\psi_n) = \min\{\overline{e}_n, \overline{o}_n\} = \left\llbracket \dfrac{F_{n+1} - 1}{2} \right\rrbracket$

2. $\alpha(\psi_n) = \max\{\overline{e}_n, \overline{o}_n\} = \left\llbracket \dfrac{F_{n+1}}{2} + 1 \right\rrbracket$

PROOF. Let Y be an independent set of edges of ψ_n. For any $v \in \overline{E}_n$, there is at most one edge in Y passing through v. Similarly, for any $u \in \overline{O}_n$, there is at most one edge in Y passing through u. Thus, $|Y|$ cannot exceed $\overline{e}_n$ and $\overline{o}_n$, and so $\alpha'(\psi_n) \leq \min\{\overline{e}_n, \overline{o}_n\}$. To establish equality, it suffices to construct an independent set of edges containing exactly $\min\{\overline{e}_n, \overline{o}_n\}$ elements.

First, suppose $n \not\equiv 2 \mod 3$. Then, ψ_n has a Hamiltonian path and such a path must alternate between vertices in $\overline{E}_n$ and $\overline{O}_n$. Thus, taking the first edge of the Hamiltonian path and every other edge after that in the path, we form an independent set of edges whose cardinality is $\min\{\overline{e}_n, \overline{o}_n\}$. Moreover, when $n \not\equiv 2 \mod 3$, $|\overline{e}_n - \overline{o}_n| = 1$. From $\overline{e}_n + \overline{o}_n = F_{n+1}$, it follows that $\min\{\overline{e}_n, \overline{o}_n\} = \frac{F_{n+1}-1}{2}$. Since F_{n+1} is odd whenever $n \not\equiv 2 \mod 3$, this quantity is equal to $\left\llbracket \frac{F_{n+1}-1}{2} \right\rrbracket$.

Suppose $n \equiv 2 \mod 3$. Then $|\overline{e}_n - \overline{o}_n| = 2$, and therefore $\min\{\overline{e}_n, \overline{o}_n\} = \frac{F_{n+1}-2}{2}$. Recall that the vertices of ψ_n can be partitioned into those of $10\Gamma_{n-2}$ and $0\Gamma_{n-3}01$, and that $|V(\psi_n)| = F_{n+1}$, $|V(\Gamma_{n-2})| = F_n$ and $|V(\Gamma_{n-3})| = F_{n-1}$. When $n \equiv 2 \mod 3$, F_{n+1} is even, while F_n and F_{n-1} are both odd. In [4], it was proved that a Fibonacci cube with odd order has a cycle containing all its vertices except one. These cycles in $10\Gamma_{n-2}$ and $0\Gamma_{n-3}01$ have lengths $F_n - 1$ and $F_{n-1} - 1$, respectively. By taking every other edge from each of these cycles, a set of $\frac{F_n - 1}{2} + \frac{F_{n-1}-1}{2} = \frac{F_{n+1}-2}{2}$ independent edges is formed. Therefore, indeed, $\alpha'(\psi_n) = \frac{F_{n+1}-2}{2}$. Since F_{n+1} is even, $\frac{F_{n+1}-2}{2} = \left\llbracket \frac{F_{n+1}-1}{2} \right\rrbracket$. This proves the first statement.

Let us now establish the result for $\alpha(\psi_n)$. Assume $\overline{e}_n < \overline{o}_n$. Now, $\alpha(\psi_n) \geq \overline{o}_n$, since $\overline{O}_n$ is an independent set. By the previous result, ψ_n has $\overline{e}_n$ independent edges. From the bipartition of $V(\psi_n)$, it follows that every vertex v_e in $\overline{E}_n$ can be paired with a vertex v_o in $\overline{O}_n$, such that the edges formed by this matching are independent. Thus, a set I of independent vertices must contain only at most one of v_e and its corresponding v_o. This implies that $|I|$ cannot exceed $\overline{o}_n$. Hence, $\alpha(\psi_n) = \overline{o}_n$. The same argument shows that if $\overline{o}_n < \overline{e}_n$, then $\alpha(\psi_n) = \overline{e}_n$. Therefore, $\alpha(\psi_n) = \max\{\overline{e}_n, \overline{o}_n\} = F_{n+1} - \left\llbracket \frac{F_{n+1}-1}{2} \right\rrbracket = \left\llbracket \frac{F_{n+1}}{2} + 1 \right\rrbracket$. □

4. SUMMARY AND COMPARISON

By tweaking the condition on the vertices of the Fibonacci cube and the Lucas cube, we constructed a new graph, the Lucci cube, which curiously also has a Fibonacci number as its order. See Table 2 [4, 9, 10] for a comparison of Γ_n, Λ_n, and ψ_n.

G	Γ_n	Λ_n	ψ_n
$\lvert V(G)\rvert$	F_{n+2}	L_n	F_{n+1}
$\lvert E(G)\rvert$	$\frac{nF_{n+1}+2(n+1)F_n}{5}$	nF_{n-1}	$\frac{(n+4)F_n+2(n-5)F_{n-1}}{5}$
rad(G)	$\llbracket \frac{n+1}{2} \rrbracket$	$\llbracket \frac{n+1}{2} \rrbracket$	$\llbracket \frac{n}{2} \rrbracket$
$Z(G)$	$^{*}\begin{cases} \{(0)^n\} \\ \{(0)^n, U_n\}^{**} \end{cases}$	$\{(0)^n\}$	$\{1(0)^{n-2}1\}$
diam(G)	n	$^{*}\begin{cases} n \\ n-1 \end{cases}$	$^{*}\begin{cases} n \\ n-1 \end{cases}$
HP***	$n \geq 1$	$n \not\equiv 0 \mod 3$	$n \not\equiv 2 \mod 3$
$\alpha'(G)$	$\left\llbracket \frac{F_{n+2}}{2} \right\rrbracket$	$\llbracket \frac{L_n - 1}{2} \rrbracket$	$\left\llbracket \frac{F_{n+1}-1}{2} \right\rrbracket$
$\alpha(G)$	$\left\llbracket \frac{F_{n+2}+1}{2} \right\rrbracket$	$\llbracket \frac{L_n}{2} + 1 \rrbracket$	$\left\llbracket \frac{F_{n+1}}{2} + 1 \right\rrbracket$

* Here, the first case holds when n is even; the second when n is odd.
**$U_n = (0)^{\frac{n-1}{2}} 1 (0)^{\frac{n-1}{2}}$
*** Existence of Hamiltonian path

Table 2: Comparison of properties of the Fibonacci, Lucas, and Lucci cubes

For future work, finding the degree sequence of the Lucci cube as in [7] and generalizing Lucci cubes as in [5, 6] may be worthwhile endeavors. One may also want to measure the observability of ψ_n (see [3]). And one could seek to relate the graph to the resonance graph of some hydrocarbon, as done in [8, 11].

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