On the Translations of an Autograph

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ABSTRACT

A graph G is an autograph if its vertices can be labeled bijectively by a multiset S of numbers called signature such that two vertices are adjacent if and only if the absolute difference of the corresponding labels is also in S. Given a signature S and its corresponding autograph $G(S)$, the autograph with signature whose elements resulted from adding a fixed real number to every element of S is called a translation of $G(S)$. In this study, properties of translations of autographs were determined. These include the number of edges a translation consists and some necessary conditions for two translations to be isomorphic. The exact number of nonempty translations of an autograph with signature consisting of integers was also found. This result is a refinement of the previous one which only gives bounds on the number of nonempty translations an autograph could have.

Keywords

autograph, signature, translation

1. INTRODUCTION

The concept of autograph was introduced by Bloom et al. [1] in 1979. A graph G is an *autograph* if its vertices can be labeled bijectively by a multiset S of real numbers such that two vertices are adjacent if and only if the absolute difference of the corresponding labels is in S. The multiset S of numbers is commonly known as a *signature* of the autograph $G(S)$ and each of its elements is called a *signature value*. In 1982, Gervacio [4] adapted this labeling principle to directed graphs, calling the resulting graphs as *difference digraphs*. In 2009, Hegde and Vasudeva [7] introduced *mod difference digraphs* using modulo difference as the criterion for the adjacency and were able to show in [8] that some structural properties of directed graphs can be studied in the context of mod difference digraphs. Since the introduction of autographs, most works done focused on identifying which graphs are autographs [1, 5, 9, 12, 3], difference digraphs [2, 10, 11] or mod difference digraphs [6, 7, 8]. The works of Panopio in [9] and Hegde and Vasudeva in [8], however, suggest that the study of this labeling method can be done through a different approach, that is, to investigate the properties of the labeled graph through observation of its signature.

2. PRELIMINARIES AND DEFINITIONS

Let $v_1, v_2, \ldots, v_n$ be n distinct vertices of a graph G and $S = \{s_1, s_2, \ldots, s_n\}$ be a multiset of real numbers. For each i, assign the value s_i to the vertex v_i. The assigned value s_i to v_i will now be called as the signature value of v_i. Two vertices v_i and v_j, $i \neq j$, are adjacent if and only if $|s_i - s_j| \in S$. Then the set $V = \{v_1, v_2, \ldots, v_n\}$ together with the determined edges is an autograph. Now observe that if $S' = \{s'_1, s'_2, \ldots, s'_n\}$ is another multiset of numbers that bijectively corresponds to set V, then $G(S')$ is also an autograph with vertex set V but with different edge set. This has been the motivation of the following definition.

Definition 2.1. (*R. G. Panopio,* [9]). *If $S = \{s_1, s_2, \ldots, s_n\}$ is a multiset of real numbers and $x \in \mathbb{R}$, the translation of S by x is the multiset $S + x = \{s_1 + x, s_2 + x, \ldots, s_n + x\}$.*

The autograph having $S + x$ as a signature is called a *translation* of the autograph $G(S)$ denoted by $G(S + x)$. See Figure 1 for illustration.

Panopio in [9] has given a necessary and sufficient condition for a translation of an autograph to be nonempty. With this, he was able to show that the number of values of x for which $G(S + x)$ is nonempty is finite.

Theorem 2.1. (*R. G. Panopio,* [9]). *Let $S = \{s_1, s_2, \ldots, s_n\}$ be a multiset of real numbers and $x \in \mathbb{R}$. Then the autograph $G(S + x)$ has at least one edge if and only if there exist i, j, k, where $i \neq j$, such that $|s_i - s_j| = s_k + x$.*

In the same paper, Panopio also obtained bounds for the number of values that can be taken by x such that $G(S + x)$ is nonempty.

Theorem 2.2. *(R. G. Panopio, [9]). Let $S = \{s_1, s_2, \ldots, s_n\}$ be a multiset of real numbers and define*

$$x' = \max_{i\neq j} |s_i - s_j| - \min_k s_k \text{ and}$$
$$x'' = \min_{i\neq j} |s_i - s_j| - \max_k s_k.$$

Then $G(S + x)$ is the empty graph if $x > x'$ or $x < x''$.

It will be apparent in the succeeding discussion that x' and x'' of Theorem 2.2 are the maximum and the minimum values of x, respectively, such that $G(S + x)$ is nonempty.

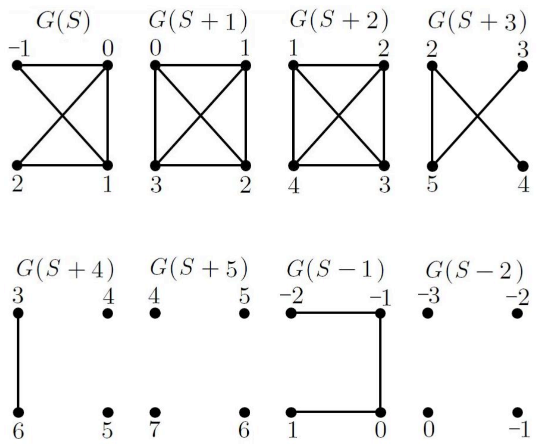

Figure 1: Autograph $G(S)$ with $S = \{-1, 0, 1, 2\}$ and some of its translations: $G(S + x)$ where $x = 1, 2, 3, 4, 5, -1, -2$.

For brevity, in this paper, vertices and their corresponding signature values are treated in a similar manner i.e., a signature value refers to the vertex it represents. For instance, for $s_i, s_j \in S$, when s_i is said to be adjacent to s_j it means that not only $|s_i - s_j| \in S$ but corresponding vertices v_i and v_j are also adjacent. Moreover, an edge $[v_i, v_j]$ will also sometimes be referred to as edge $[s_i, s_j]$. The following definition, adapted from a concept given in [8], will be vital in the later discussion.

Definition 2.2. *A vertex s in autograph $G(S)$ is called a working vertex if there exist $s_i, s_j \in S$ such that $|s_i - s_j| = s$. Further, the edge $[s_i, s_j]$ is said to correspond to s.*

3. NUMBER OF EDGES OF A TRANSLATION

Let G be an autograph with signature S of order n and $R = \{|s_i - s_j| | s_i, s_j \in S \text{ and } i \neq j\}$. Also, let $\overline{R} = \{\overline{s_i} | i = 1, 2, \ldots, m \text{ for some } m\}$ be the set of all distinct elements from R. Finally, for each $i = 1, 2, \ldots, m$, let k_i be the number of occurrences of $\overline{s_i}$ in R. The following results give the number of edges of a translation of an autograph based on the occurences of the elements of its signature in R.

Theorem 3.3. *Let S be a multiset and $x \in \mathbb{R}$. Suppose $S + x$ contains elements from $\overline{R}$ say $\overline{s_1}, \overline{s_2}, \ldots, \overline{s_r}$, $r \leq m$. Then $G(S + x)$ has exactly $k_1 + k_2 + \cdots + k_r$ number of edges.*

Proof. Suppose $\overline{s_t} \in \overline{R}$. Then there exist $s_i, s_j \in S$ such that $|s_i - s_j| = \overline{s_t}$, implying that $|(s_i + x) - (s_j + x)| = \overline{s_t}$. Suppose also that $\overline{s_t}$ is contained in $S + x$. Hence, there are corresponding vertices $v_i, v_j \in V(G(S + x))$ which form an edge in $G(S + x)$. Since $\overline{s_t}$ occurs k_t times in R, there are other $k_t - 1$ pairs of vertices that also form $k_t - 1$ edges in $G(S + x)$. Now let $\overline{s_1}, \overline{s_2}, \ldots, \overline{s_r}$ be all the elements in $\overline{R}$ that are contained in $S + x$. From the previous argument, it follows that $G(S + x)$ has at least $k_1 + k_2 + \cdots + k_r$ edges.

Suppose that there is another edge say $[v_p, v_q]$ in $G(S + x)$ aside from the edges described above. Then there exist corresponding $s_p, s_q \in S$ such that $|(s_p + x) - (s_q + x)| \in S + x$ and $|(s_p + x) - (s_q + x)| = |s_p - s_q| \in \overline{R}$. This indicates that $[v_p, v_q]$ has been accounted already. It follows, therefore, that $G(S + x)$ has exactly $k_1 + k_2 + \cdots + k_r$ edges. ∎

Corollary 3.1. *Let x' and x'' be the numbers described in Theorem 2.2. Also, let k_{max} and k_{min} be the number of occurrences of $\max_{i\neq j} |s_i - s_j|$ and $\min_{i\neq j} |s_i - s_j|$ in R, respectively. Then $G(S+x')$ and $G(S + x'')$ will have exactly k_{max} and k_{min} number of edges, respectively.*

Proof. (The following will show that $G(S + x')$ will have exactly k_{max} number of edges, the proof to show the other case can be done similarly.)

Clearly, $S + x'$ contains $\min_k s_k + x'$ but $x' = \max_{i\neq j} |s_i - s_j| - \min_k s_k$. Hence, $S + x'$ has $\max_{i\neq j} |s_i - s_j|$ as an element. So by Theorem 3.3, $G(S + x')$ have at least k_{max} number of edges.

Let $s_i \in S$ be such that it is not the minimum signature value. Thus $s_i + x' = s_i + (\max_{i\neq j} |s_i - s_j| - \min_k s_k) = (s_i - \min_k s_k) + \max_{i\neq j} |s_i - s_j| > \max_{i\neq j} |s_i - s_j|$. This means that $s_i + x' \notin R$. Thus $G(S + x')$ will not contain any other edges except the k_{max} edges given above. The result follows. ∎

Corollary 3.2. *If the elements of S are distinct, $G(S + x')$ will only have one edge.*

Proof. If the elements of S are distinct, $\max_k s_k$ and $\min_k s_k$ are also distinct. Thus $\max_{i\neq j} |s_i - s_j| = \max_k s_k - \min_k s_k$ will also be distinct and will occur just once in R. Consequently, $G(S + x')$ will only have one edge. ∎

Observe that even if the elements of S are distinct, it does not follow that $G(S + x'')$ will only have one edge, for $\min_{i\neq j} |s_i - s_j|$ could occur several times in R.

4. ISOMORPHIC TRANSLATIONS

Note that Theorem 2.1 is equivalent to the statement that $G(S+x)$ is nonempty if and only if $S+x$ contains at least one element from $R = \{|s_i - s_j||\ s_i, s_j \in S \text{ and } i \neq j\}$. The following are some consequences of this result showing ways to determine isomorphisms among translations of an autograph.

THEOREM 4.4. *Let $S = \{s_1, s_2, \ldots, s_n\}$ be a multiset and $x_1, x_2 \in \mathbb{R}$. If $S + x_1$ contains elements from $R = \{|s_i - s_j||s_i, s_j \in S$ and $i \neq j\}$ which are also contained in $S + x_2$, then $G(S + x_1)$ and $G(S + x_2)$ are isomorphic.*

Proof. If $S + x_1$ and $S + x_2$ both have no elements from R, then by Theorem 2.1, $G(S + x_1)$ and $G(S + x_2)$ are both empty which are consequently isomorphic since $|S| = |S + x_1| = |S + x_2|$.

Now suppose $S + x_1$ and $S + x_2$ contain the same elements from R. Note that $S + x_1$ and $S + x_2$ represent the vertex sets of $G(S + x_1)$ and $G(S + x_2)$, respectively. Define the mapping $f : S + x_1 \to S + x_2$ such that $f(s') = s' - (x_1 - x_2)$. Since $s' = s + x_1$ for some $s \in S$, $f(s') = f(s + x_1) = s + x_2$. Thus $s + x_1 \mapsto s + x_2$ for all $s \in S$. Clearly, f is a bijection from $S + x_1$ to $S + x_2$.

Let $[s_i', s_j']$ be an edge in $G(S + x_1)$ such that $s_i' = s_i + x_1$ and $s_j' = s_j + x_1$. This implies that $|s_i' - s_j'| = |(s_i + x_1) - (s_j + x_1)| = |s_i - s_j| \in S + x_1$ and $|s_i' - s_j'| \in R$. Notice that $|f(s_i') - f(s_j')| = |[s_i' - (x_1 - x_2)] - [s_j' - (x_1 - x_2)]| = |s_i' - s_j'| = |s_i - s_j|$. By the assumption, $|s_i - s_j| \in S + x_2$. Hence $[f(s_i') - f(s_j')]$ must be an edge in $G(S + x_2)$. Therefore, $G(S + x_1)$ and $G(S + x_2)$ are isomorphic. ■

THEOREM 4.5. *Let $S = \{s_1, s_2, \ldots, s_n\}$ be a multiset and $x_1, x_2 \in \mathbb{R}$ and suppose that for each $i = 1, 2, \ldots, n$, $|s_i - s_j| \neq |s_i - s_k|$ for all $j, k \neq i$. If each of $S + x_1$ and $S + x_2$ contains only one element from R such that these elements are of the same number of occurrences in R, then $G(S + x_1)$ and $G(S + x_2)$ are isomorphic.*

Proof. Let $\overline{s_i}, \overline{s_j} \in R$ and suppose that $S + x_1$ and $S + x_2$ contain $\overline{s_i}$ and $\overline{s_j}$, respectively, such that these elements are of the same number of occurrences in R. Also, suppose that these elements are the only elements from R that $S + x_1$ and $S + x_2$ contain. Then from Theorem 3.3, $G(S + x_1)$ and $G(S + x_2)$ have the same number of edges.

Assuming that for each $i = 1, 2, \ldots, n$, $|s_i - s_j| \neq |s_i - s_k|$ for all $j, k \neq i$, then $|(s_i + x_r) - (s_j + x_r)| \neq |(s_i + x_r) - (s_k + x_r)|$ for all $x_r \in \mathbb{R}$. This implies that each of $[s_i + x_r, s_j + x_r]$ and $[s_i + x_r, s_k + x_r]$ corresponds to a distinct working vertex in $G(S + x_r)$, that is, there is no two adjacent edges in $G(S + x_r)$ that correspond to the same working vertex. From this argument, it follows that the edges of $G(S + x_1)$ are pairwise vertex disjoint. Similarly, the edges of $G(S + x_2)$ are also pairwise vertex disjoint. So, each of $G(S + x_1)$ and $G(S + x_2)$ can be portitioned into two subgraphs. The first subgraph consists of isolated vertices while the second subgraph is a matching. Since $|S| = |S + x_1| = |S + x_2|$ and $|E(G(S + x_1))| = |E(G(S + x_2))|$, the matchings in $G(S + x_1)$ and in $G(S + x_2)$ must be isomorphic. It is necessary then that the subgraphs of isolated vertices of $G(S + x_1)$ and $G(S + x_2)$ are also isomorphic. Therefore, $G(S + x_1)$ is isomorphic to $G(S + x_2)$ ■

5. NONEMPTY TRANSLATIONS

Theorem 2.1 provides a necessary and sufficient condition for a translation of an autograph to be nonempty. Theorem 2.2, on the other hand, gives bounds on the values that can be taken by x so that $G(S+x)$ is nonempty. The results below, however, give the exact number of nonempty translations of an autograph with signature consisting of integers.

THEOREM 5.6. *If a signature S is composed of distinct integers, then there is no non-integer x such that $G(S + x)$ is nonempty.*

Proof. If S is composed of distinct integers then $R = \{|s_i - s_j||s_i, s_j \in S$ and $i \neq j\}$ will, clearly, be composed of non-zero integers. On the other hand, in order for $G(S + x)$ to be nonempty, $S + x$ must at least contain an element from R. If x is non-integer, all elements of $S + x$ will be non-integer. Thus, $S + x$ will not contain any element from R and so $G(S + x)$ will be empty if x is non-integer. ■

THEOREM 5.7. *Let S be a signature of order n composed of integers. Also, let $\overline{R} = \{\overline{s_i}|i = 1, 2, \ldots, m \text{ for some } m\}$ be the set of all distinct elements from $\{|s_i - s_j||s_i, s_j \in S$ and $i \neq j\}$. Suppose s_0 is the smallest element of S and $M = \{\overline{s_i} \in \overline{R}|\overline{s_i} > s_0\}$. Then the number of positive integer x such that $G(S + x)$ is nonempty is given by $|M|$.*

Proof. Let s_0 be the smallest element of S. Also, let $\overline{s_i} \in \overline{R}$ be such that $\overline{s_i} > s_0$. Since $\overline{s_i}$ is a positive integer, there is a positive integer x_i such that $s_0 + x_i = \overline{s_i}$. Consequently, $G(S + x_i)$ will not be empty. Let $M = \{\overline{s_i} \in \overline{R}|\overline{s_i} > s_0\}$. Since elements of M are distinct and are all positive, for each $\overline{s_i} \in \overline{R}$ there exists a unique positive integer x_i such that $s_0 + x_i = \overline{s_i}$. Hence, there are $|M|$ number of positive $x's$ such that $G(S + x)$ is nonempty. ■

The following result can be proven similarly.

THEOREM 5.8. *Let S be a signature of order n composed of integers. Also, let $\overline{R} = \{\overline{s_i}|i = 1, 2, \ldots, m \text{ for some } m\}$ be the set of all distinct elements from $\{|s_i - s_j||s_i, s_j \in S$ and $i \neq j\}$. Suppose s_l is the largest element of S and $N = \{\overline{s_i} \in \overline{R}|\overline{s_i} < s_l\}$. Then the number of negative integer x such that $G(S + x)$ is nonempty is given by $|N|$.*

Therefore, given an autograph G with signature S composed of integers, the exact number of nonempty translations of $G(S)$ can now be determined. This observation is stated in the following theorem.

THEOREM 5.9. *Let S be a signature of order n composed of integers. Also, let $\overline{R} = \{\overline{s_i} | i = 1, 2, \ldots, m \ for \ some \ m\}$ be the set of all distinct elements from $\{|s_i - s_j| | s_i, s_j \in S$ and $i \neq j\}$. Suppose s_0 and s_l are the smallest and the largest element of S, respectively. Set $M = \{\overline{s_i} \in \overline{R} | \overline{s_i} > s_0\}$ and $N = \{\overline{s_i} \in \overline{R} | \overline{s_i} < s_l\}$. Then the number of non-zero integer x such that $G(S + x)$ is nonempty is given by $|M| + |N|$.*

Proof. See the combined proofs of Theorems 5.7 and 5.8 ∎

6. SUMMARY

In this study, the number of edges a translation of an autograph consisted was determined. Some necessary conditions telling when translations of an autograph are isomorphic were also obtained. Finally, a way to determine the exact number of values of x such that $G(S + x)$ is nonempty or the exact number of nonempty translations of $G(S)$, where S consists of integral elements, was found. It is recommended that further studies be conducted on translations of autographs to determine other properties these graphs hold.

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