An Algorithm for Propagating Graceful Trees Using the Adjacency Matrix of Given Graceful Trees

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ABSTRACT

A tree T on $n+1$ vertices is said to be graceful if its vertices can be labeled using all of the integers $0, 1, \ldots, n$ such that the edge weights will run from 1 through n, and where the weight of an edge equals the absolute value of the difference of the numbers assigned to its vertices. In this paper, we develop an algorithm that uses the adjacency matrices of known graceful trees to propagate new graceful trees through duplication of arbitrary graceful trees and joining them by additional edges. The algorithm developed facilitates faster generation of families of graceful trees, the study of which is one of the current directions in establishing the graceful tree conjecture.

1. PRELIMINARIES AND DEFINITIONS

Let T be a tree on $n+1$ vertices with vertex set $V(T)$ and edge set $E(T)$. A graceful valuation g of T is an injection

$$g : V(T) \to \{0, 1, \ldots, n\}$$

such that if

$$w(u, v) = |g(u) - g(v)| \ \forall (u, v) \in E(T),$$

then

$$w : E(T) \to \{1, 2, \ldots, n\}$$

is an injection. A tree which admits a graceful valuation is said to be *graceful*.

A simple graph that is connected with $n+1$ vertices and n edges is a tree [2].

A graph G is said to be *bipartite* if $V(G)$ can be partitioned into sets $A_G = \{x_1, x_2, \ldots, x_p\}$ and $B_G = \{y_1, y_2, \ldots, y_q\}$ such that $(x_i, x_j) \notin E(G)$ for all $1 \leq i, j \leq p$ and $(y_i, y_j) \notin E(G)$ for all $1 \leq i, j \leq q$.

The *distance* $d(u, v) = d(v, u)$ between any two vertices u and v in a connected graph G is the length of the shortest path between u and v.

The *parity set* $P(v)$ of a vertex v in a tree T is the set of all vertices including v which are of even distance from v in T. The *base* of T under its graceful valuation g is that vertex b in T such that $g(b) = 0$.

A graceful valuation g is *interlaced* if g induces, by restriction, a bijection between $(P(b)$ and $N_{s-1} = \{0, 1, 2, ..., s-1\}$ where s, the *size* of T, is the number of vertices of T in $P(v)$.

To illustrate the above-mentioned definitions, consider the tree T shown in Figure 1.1 with its graceful valuation g in Figure 1.2.

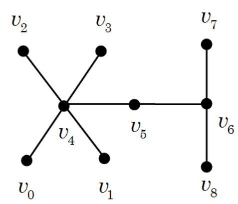

Figure 1.1

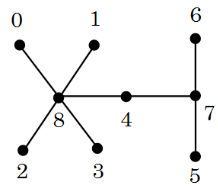

Figure 1.2

In Figure 1.2, it follows that the base of T under its graceful valuation g is v_2 and from Figure 1.1,

$$P(v_2) = \{v_0, v_1, v_2, v_3, v_5, v_7, v_8\}.$$

Clearly, $P(v_2)$ has $s = 7$ elements so that

$$N_{s-1} = N_6 = \{0, 1, 2, 3, 4, 5, 6\}.$$

Now, g is interlaced since it induces a bijection between $P(v_2)$ and N_6 as illustrated in Figure 1.3.

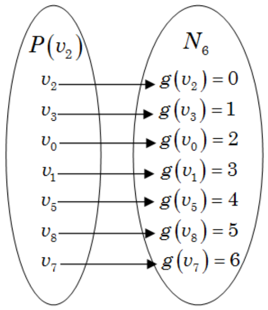

Figure 1.3

The adjacency matrix A_T of a tree T with vertex set $V(T) = \{v_0, v_1, \ldots, v_n\}$ is the $(n+1) \times (n+1)$ matrix (a_{ij}) defined by

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E(T) \\ 0 & \text{otherwise} \end{cases} .$$

In general, if the vertices of a graceful tree are identified with their labels in the graceful valuation of the tree, then the adjacency matrix can be written in such a way that the sum of the diagonal elements except for the main diagonal is 1. This is because each element of a diagonal of the said matrix corresponds to a representative class of edges of the same weight. Since an edge weight in a graceful tree is unique, there can only be one such edge. Thus, the adjacency matrix of any graceful tree can be written in such form which is called *generalized adjacency matrix* of a tree induced by its graceful valuation.

2. BACKGROUND OF THE STUDY

In 1963, Kotzig and Ringel conjectured that all trees are graceful [1]. Since then, various efforts have been made to settle the validity of this conjecture but to date, it has remained unresolved. A recent survey of graceful trees can be found in Gallian [3] which includes paths, stars, caterpillars, olive trees, banana trees, and symmetrical trees, among others.

Hugo [4] used the adjacency matrix of a graceful graph to develop two algorithms to generate all possible extensions of the graph to yield graceful graphs. His first algorithm requires the addition of a vertex or an edge by augmenting two diagonals with exactly one “1” entry. The propagation of graceful graphs is done recursively. His second algorithm requires the addition of any number of vertices or edges by augmenting any number of rows or columns such that there is exactly one “1” entry in each diagonal.

In this paper, we give an algorithm for propagating graceful trees using the adjacency matrices of graceful trees. The algorithm developed may be used to generate graceful trees in order to identify properties possessed by graceful trees. Attempts to directly prove the Graceful Tree Conjecture have all failed. Proving the conjecture indirectly will start by assuming that there is a tree which is not graceful so it is not isomorphic to any member of a known family of graceful trees. A contradiction must occur somewhere to prove the conjecture and this is where properties of a family of graceful trees could come in.

3. THE ALGORITHM

The following theorem will be needed in our construction in the Algorithm.

THEOREM 3.1. *Let T_i for $i = 1, \ldots, m$ be m disjoint graceful trees on n_i+1 vertices with adjacency matrix $A_{T_i} = [a_{ij}]$. Then the matrix*

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \ldots & 0 & A_{T_1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & A_{T_2} & 0 \\ 0 & 0 & \ddots & & \ldots & \ldots & 0 & \vdots \\ 0 & & & 0 & A_{T_m} & \ldots & & 0 \\ 0 & & \ldots & A_{T_m} & 0 & & & 0 \\ \vdots & & \ldots & \ldots & & \ddots & 0 & 0 \\ 0 & A_{T_2} & 0 & \ldots & 0 & 0 & 0 & 0 \\ A_{T_1} & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with dimension

$$2(n_1 + n_2 + \cdots + n_m + m) \times 2(n_1 + n_2 + \cdots + n_m + m)$$

and with 2 submatrices of A_{T_i}, $i = 1, 2, \ldots, m$ where the rest of the entries of A are zero, is an adjacency matrix for the disjoint union of 2 copies of T_i, $i = 1, 2, \ldots, m$.

Proof. By construction, A is a $(0, 1)$-matrix and since A_{T_i}'s are symmetric, so is A. Also, the main diagonal of A has all zero entries since the main diagonal does not contain entries from any submatrix A_{T_i}. So, A is an adjacency matrix.

Consider $V(T_i)$ to be the vertex set of T_i, $i = 1, 2, \ldots, m$ such that $|V(T_i)| = n_i + 1$. Let $x_i \in V(T_i)$ be the base of T_i under the graceful valuation f_i. Since T_i is a bipartite, we can fix a bipartition V_{i1} and V_{i2} of the vertices of T_i such that $x_i \in V_{i1}$. Moreover,

$$V_{i1} = \{v \in V(T_i) | d(x_i, v) \textit{ is even}\}$$

and

$$V_{i2} = \{v \in V(T_i) | d(x_i, v) \textit{ is odd}\}.$$

In the generalized adjacency matrix A_{T_i} of T_i, $a_{kj} = 1$ implies that $v_k \in V_{i1}$ and $v_j \in V_{i2}$. Let C_i and D_i be the sets of labels on the vertices of V_{i1} and V_{i2}, respectively.

Now consider the $2m$ copies of adjacency matrices as assembled in A. Call the copy of adjacency matrix as the j^{th} copy if it is the j^{th} counting from the bottom left corner of A. For a set of integers S, let $a + S = \{a + s | s \in S\}$ for any $a \in \mathbb{Z}$. The 1^{st} copy of A_{T_i} in A defines edges between vertices with labels $C_1 \cup (\hat{n} + D_1)$ where

$$\hat{n} = 2\sum_{i=1}^{m}(n_i + 1) - (n_1 + 1).$$

There is an edge $v_s v_{\hat{n}+q}$ for some $s \in C_j$ and $q \in D_j$ precisely when $v_s \in V_{i1}$ and $v_q \in V_{i2}$ and $v_s v_q \in E(T_i)$. Thus, the 1^{st} copy of adjacency matrix in A that represents T_1 induces an isomorphic copy of T_1 on the corresponding vertex set.

For $j \in \{2, \dots, 2m\}$, the j^{th} copy of adjacency matrix in A defines edges between vertices with labels $(\overline{n}+C_j) \cup (\hat{n}+D_j)$ where

$$\overline{n} = \sum_{i=1}^{j-1} (n_i + 1)$$

and

$$\hat{n} = 2\sum_{i=1}^{m} (n_i + 1) - \sum_{i=1}^{j} (n_i + 1).$$

There is an edge $v_{\overline{n}+s} v_{\hat{n}+q}$ for some $s \in C_j$ and $q \in D_j$ precisely when $v_s \in V_{i1}$ and $v_q \in V_{i2}$ and $v_s v_q \in E(T_i)$. Thus, the j^{th} copy of adjacency matrix in A that represents T_i induces an isomorphic copy of T_i on the corresponding vertex set.

In general, since matrix A is symmetric, the j^{th} copy and the $[2m - j + 1]^{th}$ copy are equal and that they represent the graceful tree T_j, $j = 1, 2, \dots, m$. Hence, the j^{th} and the $[2m - j + 1]^{th}$ adjacency matrices both induce isomorphic copies of T_j on their corresponding vertex sets. Thus we have, 2 copies of T_j.

Further, by construction of matrix A, the 2 copies of T_i, $i = 1, \dots, m$ are pairwise disjoint. ∎

THEOREM 3.2. *Let T_i for $i = 1, \dots, m$ be m disjoint graceful trees on $n_i + 1$ vertices with adjacency matrix $A_{T_i} = [a_{ij}]$. Then the matrix*

$$A_{T_i} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & A_{T_1} \\ 0 & 0 & 0 & \dots & & \cdots & 0 \\ 0 & 0 & \ddots & & A_{T_{m-1}} & \dots & \vdots \\ 0 & \vdots & & A_{T_m} & & & 0 \\ \vdots & \dots & A_{T_{m-1}} & & \ddots & 0 & 0 \\ 0 & \dots & & \dots & 0 & 0 & 0 \\ A_{T_1} & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix}$$

with dimension $[2(n_1 + \cdots + n_{m-1} + m - 1) + (n_m + 1)] \times [2(n_1 + \cdots + n_{m-1} + m - 1) + (n_m + 1)]$ and with 2 submatrices of A_{T_i}, $i = 1, 2, \dots, m - 1$ and a submatrix of A_{T_m}, where the rest of the entries of A are zero, is an adjacency matrix for the disjoint union of 2 copies of T_i, $i = 1, 2, \dots, m - 1$ and a copy of T_m .

The proof of Theorem 3.2 is similar with the proof of Theorem 3.1.

The Algorithm

1. Consider m disjoint graceful trees T_i, $i = 1, \dots, m$ with $n_i + 1$ vertices.

2. Use the graceful labeling of T_i to write its adjacency matrix as

$$A = \begin{pmatrix} 0 & a_{01} & a_{02} & \cdots & a_{0n} \\ a_{10} & 0 & a_{12} & \cdots & a_{1n} \\ a_{20} & a_{21} & \ddots & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n0} & a_{n1} & a_{n2} & \cdots & 0 \end{pmatrix}$$

   The sum of the c^{th} diagonal elements of A_{T_i} is 1 for each $1 \leq c \leq n$.

3. Assemble 2 copies of each A_{T_i} for $i = 1, \dots, m$ as in Theorem 3.1 to form the transition matrix

$$T(A) = [b_{ij}] = \begin{pmatrix} 0 & \cdots & & & \cdots 0 & A_{T_1} \\ \vdots & \ddots & \dots & & & 0 \\ & & 0 & A_{T_m} & & \\ & \dots & A_{T_m} & 0 & & \\ 0 & & & \dots & \ddots & \vdots \\ A_{T_1} & 0 & \cdots & & \cdots & 0 \end{pmatrix}$$

   with dimension

   $2(n_1 + n_2 + \cdots + n_m + m) \times 2(n_1 + n_2 + \cdots + n_m + m)$

   .

4. For the diagonals of $T(A)$ which are also the main diagonals of each A_{T_i} , replace exactly one "0" entry in the main diagonal of A_{T_i} by "1".

5. For each $i = 1, 2, \dots, m - 1$, let

$$c_i = min\{|V(T_i)|, |V(T_{i+1})|\}$$

   and $\overline{D}$ be the diagonal of $T(A)$ that passes through b_{0k} and b_{k0} where

$$k = 2\sum_{i=1}^{m} (n_i + 1) - 1.$$

   For the diagonal of $T(A)$ between A_{T_i} and $A_{T_{i+1}}$, consider the cells within c_i steps above and below $\overline{D}$. Replace exactly one "0" entry in those cells by "1". Denote the new transition matrix as $T^*(A)$.

6. Generate the graph given by $T^*(A)$ on

$$2\sum_{i=1}^{m} (n_i + 1)$$

   vertices and

$$2\sum_{i=1}^{m} n_i + (2m - 1)$$

   edges.

7. Repeat steps 4 to 6 to exhaust the possible cases in replacing exactly one "0" entry by "1" in the cells considered in Steps 4 and 5. Note that we can generate possibly

$$(n_1+1)\cdot(n_2+1)\cdot\dots\cdot(n_m+1)\cdot 2c_1\cdot 2c_2\cdot 2c_{m-1}$$

gracefully labeled graphs.

THEOREM 3.3. *Each graph generated by the Algorithm is a graceful tree.*

Proof. Let T_i be graceful trees on n_i+1 vertices for $i=1,2,\dots,m$. Let A_{T_i} be the corresponding adjacency matrix of T_i , for $i=1,2,\dots,m$. Now, the sum of the diagonal elements of each A_{T_i} is 1 for each $1\le c\le n$.

Let T^* be a graph generated by the Algorithm when applied to T_i, $i=1,2,\dots,m$. We wish to show that T^* is a tree. We now describe the construction of T^*. Notice that the transition matrix $T(A)=[b_{ij}]$ is constructed as in Theorem 3.1. Hence, it induces disjoint union of 2 copies of T_i, $i=1,2,\dots,m$. Note that $T(A)$ has dimension

$$2(n_1+n_2+\cdots+n_m+m)\times 2(n_1+n_2+\cdots+n_m+m)$$

and it has

$$2(n_1+n_2+\cdots+n_m)$$

"1" entries.

Now if the diagonals of $T(A)$ are also the main diagonals of each A_{T_i}, then exactly one "0" entry in the main diagonal of A_{T_i} is replaced by "1". The "1" entry in the main diagonal of A_{T_1} indicates an edge joining the 2 copies of T_1, denoted by $2T_1$, making $2T_1$ a simple connected graph of $2(n_1+1)$ vertices and $2n_1+1$ edges. Hence, $2T_1$ is a tree. Generally, the "1" entry in the main diagonal of A_{T_i}, $i=1,2,\dots,m$ indicates an edge joining the 2 copies of T_i, denoted by $2T_i$, and thereby making $2T_i$ a simple connected graph of $2(n_i+1)$ vertices and $2n_i+1$ edges. Hence, $2T_i$, $i=1,2,\dots,m$ is a tree. Consequently, there are additional m "1" entries in $T(A)$.

For each $i=1,2,\dots,m-1$, let

$$c_i=min\{|V(T_i)|,|V(T_{i+1})|\}$$

and $\overline{D}$ be the diagonal of $T(A)$ that passes through b_{0k} and b_{k0} where

$$k=2\sum_{i=1}^{m}(n_i+1)-1.$$

For the diagonal of $T(A)$ between A_{T_i} and $A_{T_{i+1}}$, we consider the cells within c_i steps above and below $\overline{D}$ and we replace exactly one "0" entry in those cells by "1". This "1" entry indicates an edge that join $2A_{T_i}$ to $2A_{T_{i+1}}$ for $i=1,2,\dots,m-1$. Consequently, there are additional $m-1$ "1" entries in $T(A)$. We denote the new transition matrix as $T^*(A)$. Observe that the graph T^* induced by $T^*(A)$ is connected with

$$2(n_1+1+n_2+1+\cdots+n_m+1)=2\sum_{i=1}^{m}n_i+2m$$

vertices and

$$2(n_1+n_2+\cdots+n_m)+m+m-1=2\sum_{i=1}^{m}n_i+2m-1$$

edges. Hence, T^* is a tree.

If a diagonal of $T(A)$ contains an element of A_{T_i}, then the sum of its elements is 1 since the sum of its elements which are in A_{T_i} is 1 and the remaining elements of the diagonal are all 0. If the diagonal of $T(A)$ is the main diagonal of A_{T_i} then the sum of its elements is 1 since exactly one of its 0 elements is replaced by 1. Similarly, if a diagonal of $T(A)$ does not contain an element of A_{T_i} then the sum of its elements is 1 since exactly one of its 0 elements is replaced by 1. Thus, T^* is a graceful tree. ■

4. APPLICATION OF THE ALGORITHM

Step 1. Consider the graceful trees T_1 and T_2 with $n_1+1=1+1=2$ vertices and $n_2+1=2+1=3$ vertices, respectively as shown in Figure 4.1

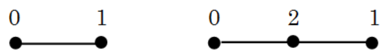

Figure 4.1

Step 2. Obtain the adjacency matrix of T_1 and T_2. See Figures 4.2.

	0	1
0	0	1
1	1	0

	0	1	2
0	0	0	1
1	0	0	1
2	1	1	0

Figure 4.2

Step 3. Assemble A_{T_1} and A_{T_2} to form the transition matrix $T(A)=[b_{ij}]$ shown in Figure 4.3

	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	0	1	0
2	0	0	0	0	0	0	0	1	0	0
3	0	0	0	0	0	0	0	1	0	0
4	0	0	0	0	0	1	1	0	0	0
5	0	0	0	0	1	0	0	0	0	0
6	0	0	0	0	1	0	0	0	0	0
7	0	0	1	1	0	0	0	0	0	0
8	0	1	0	0	0	0	0	0	0	0
9	1	0	0	0	0	0	0	0	0	0

Figure 4.3

The transition matrix $T(A)$ in Figure 4.3 is of dimension 10×10.

Step 4. For the diagonal of $T(A)$ which is also the main diagonal of A_{T_1}, replace exactly one "0" entry by "1". For the diagonal of $T(A)$ which is also the main diagonal of A_{T_2}, replace exactly one "0" entry by "1". See Figure 4.4.

	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	0	1	0	0
3	0	0	0	0	0	0	0	1	0	0
4	0	0	0	0	0	1	1	1	0	0
5	0	0	0	0	1	0	0	0	0	0
6	0	0	0	0	1	0	0	0	0	0
7	0	0	1	1	1	0	0	0	0	0
8	0	1	0	0	0	0	0	0	0	0
9	1	1	0	0	0	0	0	0	0	0

Figure 4.4

Step 5. Let

$$c_1 = min\{|V(T_1)|, |V(T_2)|\} = min\{2, 3\} = 2$$

and $\overline{D}$ be the diagonal of $T(A)$ that passes through b_{0k} and b_{k0} where $k = 2(2)+2(3)-1 = 9$. For the diagonal of $T(A)$ between A_{T_1} and A_{T_2} that does not contain an element of A_{T_1} and A_{T_2}, consider the cells within $c_1 = 2$ steps above and below $\overline{D}$. Replace exactly one "0" entry in those cells by "1" as shown in Figure 4.5.

	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	0	1	0	0
3	0	0	0	0	0	0	0	1	0	1
4	0	0	0	0	0	1	1	1	0	0
5	0	0	0	0	1	0	0	0	0	0
6	0	0	0	0	1	0	0	0	0	0
7	0	0	1	1	1	0	0	0	0	0
8	0	1	0	0	0	0	0	0	0	0
9	1	1	0	1	0	0	0	0	0	0

Figure 4.5

Step 6. Generate the tree from the given adjacency matrix in Figure 4.5 as shown in Figure 4.6. Note that if we exhaust all the possible cases by replacing exactly one "0" entry by "1" in the cells considered in Steps 4 and 5, we can generate

$$(n_1 + 1) \cdot (n_2 + 1) \cdot 2c_1 = 2 \cdot 3 \cdot 2 \cdot 2 = 24$$

possible gracefully labeled trees on 10 vertices.

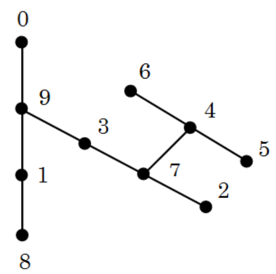

Figure 4.6

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