Solving the Optimal Three-stage Periodic Inventory Policy Using an Algorithm Coded in C♯

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ABSTRACT
In this paper we study a three-stage inventory system to come up with an optimal periodic inventory policy. We formulate a general and specific model on a three-stage serial inventory system for a single-item considering stochastic demand. To solve the model, we come up with an algorithm that can be easily implemented using C♯ as a programming language. With this algorithm, we create a user-friendly software so that even layman people can use it in their business. We observe validity of the formulated model and the effectiveness of the algorithm by generating and analyzing numerical results.

Keywords
Inventory Management, Multi-Echelon, Periodic Inventory Policy, C♯ Programming

1. INTRODUCTION
The stocks or products being offered by a company to its customers are called inventory. Management of inventory in a given company is very important and usually too complex to handle. Efficient and effective management of inventory will lessen the operating cost and, as a result, making the business more competitive in terms of finances. There are two common inventory policies being adapted by companies: the periodic inventory policy and the continuous inventory policy. These policies answer the two main questions in inventory: ”When to order?” and ”How much to order?” [3].

The main goal of finding an optimal inventory policy is to minimize the total inventory cost. There are three main cost parameters associated to the total inventory cost - the holding cost or the cost of maintaining the inventory, the ordering cost or the cost of placing an order and the shortage cost or the penalty incurred to the company when it runs out of stock [6] .

The holding cost and ordering cost are in conflict with each other. For example, if we are to order too much, the holding cost will be too high (e.g. more space to be rented, more airconditioning units needed for perishable products) but the ordering cost is low since the number of placement of orders will decrease (i.e. there will be a lesser chance of being out of stock and thus lessens the number of ordering new stocks). Whereas, when we order too small (i.e. we rent smaller space, less airconditioning units needed for perishable products) the holding cost is low but the ordering cost will be too high (i.e. there is a bigger chance of running out of stock thus more placement of orders to replenish the stocks). Note also that shortage cost is in conflict with holding cost [9], [8]. To determine the amount to order and when to place an order, we will formulate and minimize the following total inventory cost (TC)

$$TC = (OrderingCost) + (HoldingCost) + (ShortageCost).$$

Different models for a single-stage [9] and for a two-stage [5], [1] had been formulated. But there are existing inventory systems that are naturally three-stage and thus, cannot be fully explained by a single-stage or a two-stage model. An example of a three-stage inventory system is a business where products are being stored in a main warehouse (stage 3) then being distributed to main stores (stage 2) and then to retailers (stage 1) Thus, in this paper we formulate a model for a three-stage inventory system so as to minimize the total inventory cost.

In the formulation of the total inventory cost function, one of the main factors to be considered is the customer's demand. Demand can be deterministic or stochastic in nature [7]. In the formulation of the general model, we consider a stochastic demand. And in the formulation of the specific model, we consider a normally distributed demand. We solve the specific model using an algorithm to be coded in C♯ programming language. C♯ is an object-oriented programming language created by Microsoft. With the user-friendly interface generated using C♯, we create a software that will solve the formulated model.

2. RESULTS AND DISCUSSION
We formulate the general and specific model with stochastic demand. In the formulation, we will use the concept of echelon stock [2] which is the total inventory in the current stage and all its downstream stages. That is, Echelon 1 refers to Stage 1, Echelon 2 refers to Stages 2 and 1, and Echelon 3 denotes Stages 3, 2 and 1. For example, on-hand inventory

at Echelon 3 means the total on-hand inventory at Stages 3, 2 and 1.

2.1 Notations

We use the following standard notations in the formulation of the models [1].

$I_i(t)$ − on-hand inventory or the amount of inventory in the warehouse at echelon i at time t
$B_i(t)$ − amount of unfulfilled demand at echelon i at time t
$O_i(t)$ − on-order replenishment at echelon i at time t
$IL_i(t)$ − inventory level or net inventory at echelon i, at time t, equivalent to $I_i(t) - B_i(t)$
$IP_i(t)$ − inventory position at time t at echelon i, equivalent to $IL_i(t) + O_i(t)$
$TC(\cdot)$ − expected total inventory cost per unit time
L_i − replenishment leadtime at echelon i
d − expected demand per unit time
$f(x|L)$ − probability density function of leadtime demand x, given that the leadtime is L
a_i − ordering cost per cycle (per order) at echelon i
h_i − holding cost per item per unit time at echelon i
b − shortage cost per item at stage 1
T_i − review period at echelon i
R_i − order-up-to level at echelon i
n_1 − number of times Stage 1 places an order to Stage 2, for every review cycle of Stage 2
n_2 − number of times Stage 2 places an order to Stage 3, for every review cycle of Stage 3

2.2 Assumptions

The following general assumptions are considered in the study of a two-stage serial system [1] and will also be considered in this formulation. A graphical representation of these assumptions is shown in Figure 1.

A1. The cost parameters (holding cost, ordering cost and shortage cost) are quantifiable and known in advance.

A2. During the last replenishment in a cycle, Stage 2 orders the entire remaining on-hand inventory from Stage 3. Similarly, Stage 1 orders the entire on-hand inventory from Stage 2. These are equivalent to
$IP_2(L_3 + (n_2 - 1)T_2) = IL_3(L_3 + (n_2 - 1)T_2)$ and
$IP_1(L_3 + L_2 + (n_1 - 1)T_1) = IL_2(L_3 + L_2 + (n_1 - 1)T_1)$
given that Stage 3 orders at time $t = 0$.

A3. Echelon 3 has sufficient inventory to raise the Stage 2 inventory position to R_2 for all normal replenishment cycles. Similarly, Echelon 2 has sufficient inventory to raise the Stage 1 inventory position to R_1 for all normal replenishment cycles. That is, $IL_3\left(L_3 + (n_2 - 2)T_2\right) > R_2$ and $IL_2\left(L_3 + L_2 + (n_1 - 2)T_1\right) > R_1$.

A4. The inventory position in Echelon 2, shortly before the shipment from Stage 3 arrives, is less than R_2. Otherwise, Stage 3 does not need to send any shipment to Stage 2. Similarly, we will assume that the inventory position in Echelon 1, shortly before the shipment from Stage 2 arrives, is less than R_1.

A5. Each stage implements a periodic inventory policy and the ordering policies are nested. That is, Stage 2 (Stage 1) places a replenishment when Stage 3 (Stage 2) receives its replenishment. To coordinate the replenishment of both stages, the constraints $T_3 = n_2 T_2$ and $T_2 = n_1 T_1$ are imposed, where n_1 and n_2 are positive integers.

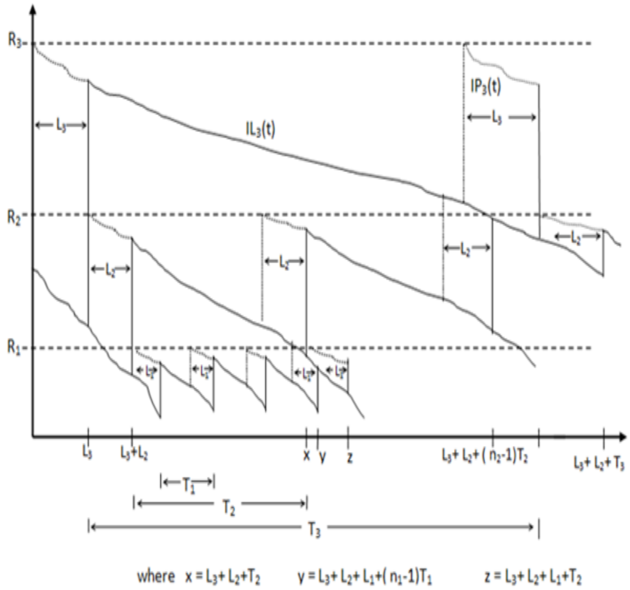

Figure 1: A three-stage inventory level and position example ($n_2 = 3, n_1 = 4$)

2.3 Derivation of the general and specific model

To formulate the cost function TC we will derive the three main cost parameters separately. Note that the ordering cost per unit time at echelon i is a_i/T_i . Thus, the expected total ordering cost per unit time is

$$\frac{a_3}{T_3} + \frac{a_2}{T_2} + \frac{a_1}{T_1} \ . \tag{1}$$

To derive the total holding cost, we will consider Echelons 3, 2 and 1 separately. The average inventory at Echelon 3 is $R_3 - d\left(L_3 + \frac{T_3}{2}\right)$. Thus, the expected holding cost per unit time at Echelon 3 is

$$h_3\left[R_3 - d\left(L_3 + \frac{T_3}{2}\right)\right]. \tag{2}$$

In the case of Echelon 2, we will derive the holding cost for the $n_2 - 1$ normal replenishment cycles and for the last (exhausted) replenishment cycle [1] (this is because of assumption A2). Note that the average inventory in Echelon 2 for every normal replenishment cycle is $R_2 - d\left(L_2 + \frac{T_2}{2}\right)$. The average inventory for the last replenishment cycle can be computed as the average of the starting and ending inventory at the last cycle. That is, the average inventory is

$$\frac{[R_3 - d(L_3 + L_2 + (n_2 - 1)T_2)] + (R_3 - d(L_3 + L_2 + T_3))}{2}$$
$$= R_3 - d\left(L_3 + L_2 + T_3 - \tfrac{T_2}{2}\right).$$

Thus, the expected holding cost per unit time at Echelon 2 is

$$h_2\left[\frac{n_2-1}{n_2}\left(R_2-d\left(L_2+\frac{T_2}{2}\right)\right)\right]+$$
$$h_2\left[\frac{1}{n_2}\left(R_3-d\left(L_3+L_2+T_3-\frac{T_2}{2}\right)\right)\right]. \quad (3)$$

Now, the expected holding cost for Echelon 1 will be analyzed like what we did in Echelon 2. Note that the average inventory for every normal replenishment cycle is $R_1 - d\left(L_1 + \frac{T_1}{2}\right)$ whereas, the average inventory in the exhaustive cycle is $R_2 - d\left(L_1 + L_2 + T_2 - \frac{T_1}{2}\right)$. Thus, the holding cost per unit time at Echelon 1 is

$$h_1\left[\frac{n_1-1}{n_1}\left(R_1-d\left(L_1+\frac{T_1}{2}\right)\right)\right]+$$
$$h_1\left[\frac{1}{n_1}\left(R_2-d\left(L_1+L_2+T_2-\frac{T_1}{2}\right)\right)\right]. \quad (4)$$

Lastly, to compute the shortage cost in Echelon 1 we will also consider the expected shortage for normal and exhaustive cycle. Note that the expected number of shortages for every normal replenishment cycle at Echelon 1 is

$\int_{R1}^{\infty}(x-R_1)\,f\left(x\,|T_1+L_1\right)dx$ while the expected number of shortages for the exhaustive cycle is $\int_{R_2}^{\infty}(x-R_2)\,f\left(x\,|T_2+L_1+L_2\right)dx$ (see [1] and [3]). Therefore, the expected shortage cost per unit time at echelon 1 is

$$\frac{b}{T_1}\left[\frac{n_1-1}{n_1}\int_{R1}^{\infty}(x-R_1)\,f\left(x\,|T_1+L_1\right)dx\right]+$$
$$\frac{b}{T_1}\left[\frac{1}{n_1}\int_{R_2}^{\infty}(x-R_2)\,f\left(x\,|T_2+L_1+L_2\right)dx\right]. \quad (5)$$

Adding equations (1), (2) ,(3), (4) and (5) gives us the general expected total inventory cost function TC. Thus, our general model is to minimize TC. Note that by assumption A5, the expected total inventory cost function can be simplified in terms of four continuous variables T_1, R_1, R_2 and R_3; and two discrete variables n_1 and n_2.

To formulate a specific model, we consider a specific distribution of demand. Let us consider a demand that is normally distributed with probability density function g defined as $g(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, where μ and σ are the mean and standard deviation of the demand, respectively.

By Tempelmeier [10], the demand during deterministic leadtime L is also normally distributed with probability density function $f\left(x\,|L\right)$ defined as,

$f\left(x\,|L\right)=\frac{1}{\sigma_Y\sqrt{2\pi}}e^{-\frac{(x-\mu_Y)^2}{2\sigma_Y^2}}$, where $\mu_Y=\mu L$ and $\sigma_Y=\sigma\sqrt{L}$.

Evaluating the integral terms in equation (5), with density function $f\left(x\,|L\right)$ as described above, will give a specific total inventory cost function where the expected shortage cost per unit time at stage 1 is given by

$$\frac{b}{T_1}\left[\frac{n_1-1}{n_1}F+\frac{1}{n_1}G\right] \quad (6)$$

where

$$F=\frac{\sigma_1 e^{-\frac{(R_1-\mu_1)^2}{2\sigma_1^2}}}{\sqrt{2\pi}}+\frac{\mu_1-R_1}{2}+\left(\frac{R_1-\mu_1}{2}\right)erf\left(\frac{R_1-\mu_1}{\sigma_1\sqrt{2}}\right),$$
$$G=\frac{\sigma_2 e^{-\frac{(R_2-\mu_2)^2}{2\sigma_2^2}}}{\sqrt{2\pi}}+\frac{\mu_2-R_2}{2}+\left(\frac{R_2-\mu_2}{2}\right)erf\left(\frac{R_2-\mu_2}{\sigma_2\sqrt{2}}\right),$$

$\mu_1=\mu\left(T_1+L_1\right),\quad \sigma_1=\sigma\sqrt{T_1+L_1},$
$\mu_2=\mu\left(L_1+L_2+T_2\right),\quad \sigma_2=\sigma\sqrt{L_1+L_2+T_2},$
$T_3=n_2T_2,\ \ T_2=n_1T_1,\ \ erf=error\,function.$

3. ALGORITHM TO SOLVE THE SPECIFIC MODEL

Note that the formulated specific model is an example of a mixed integer non-linear programming (MINLP) model. MINLP models are too complex to be solved. There are expensive softwares, such as GAMS, that are available in the market that can solve MINLP models. The interface of GAMS is not so user-friendly especially for a layman. Thus, we create a more user-friendly and a less expensive software by first constructing an algorithm that can be easily implemented using C♯ codes.

In equation (6), the error function is defined as $erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ which cannot be easily evaluated using C♯ codes but can be approximated in different ways. In the algorithm, we use the approximation formulated by Hastings [4] in which
$erf(x)\approx 1-\left(a_1t+a_2t^2+a_3t^3\right)e^{-x^2}$ where $t=\frac{1}{(1+a_4x)}$, $a_1=0.348022, a_2=-0.0958798, a_3=0.7478556$ and $a_4=0.47047$.

Combining assumptions A3 and A4, we add in the algorithm the following constraints:
$C1:\ d\left(L_3+(n_2-2)\,T_2\right)\le R_3-R_2\le d\left(L_3+T_3\right)$ and
$C2:\ d\left(L_3+L_2+(n_1-2)\,T_1\right)\le R_2-R_1\le d\left(L_3+L_2+T_2\right)$.

Since we do not want a negative holding cost per review period at every stage, we need to add also the following constraints
$C3:\ R_3-d\left(L_3+L_2+n_2T_2-\frac{T_2}{2}\right)\ge 0$
$C4:\ R_3-d\left(L_3+\frac{T_3}{2}\right)\ge 0$
$C5:\ R_2-d\left(L_2+\frac{T_2}{2}\right)\ge 0$
$C6:\ R_2-d\left(L_1+L_2+T_2-\frac{T_1}{2}\right)\ge 0$

Observe that $\frac{\partial TC}{\partial R_3}=h_3+\frac{h_2}{n_2}>0$, which means that TC increases as R_3 increases. Thus, in our algorithm, we choose the minimum R_3 that satisfies constraints C1, C3 and C4. The derived specific cost function TC can be easily shown to be convex with respect to R_1 and R_2 for fixed values of $n_1>1$, n_2, R_3 and T_1. That is, since

$$\frac{\partial^2TC}{\partial R_1^2}=\frac{\sqrt{2}b\left(n_1-1\right)e^{-\frac{(R_1-\mu(T_1+L_1))^2}{2\sigma^2(T_1+L_1)}}}{2T_1n_1\sqrt{\pi}\sigma\sqrt{T_1+L_1}}>0$$

when $n_1>1$;

$$\frac{\partial^2TC}{\partial R_2^2}=\frac{\sqrt{2}be^{-\frac{(R_1-\mu(n_1T_1+L_1+L_2))^2}{2\sigma^2(n_1T_1+L_1+L_2)}}}{2T_1n_1\sqrt{\pi}\sigma\sqrt{n_1T_1+L_1+L_2}}>0;$$

and

$$\frac{\partial^2 TC}{\partial R_2 \partial R_1} = \frac{\partial^2 TC}{\partial R_1 \partial R_2} = 0.$$

Now, we use the following algorithm to solve the specific model:

ALGORITHM

Inputs: h_1, h_2, h_3, a_1, a_2, a_3, L_1, L_2, L_3, b, μ, σ^2
Initialize n, m, p, OptimalTC, n_1, n_2, T_1, R_1, R_2 and R_3 ($n = m = 100, p = 365$, OptimalTC="infinity or relatively large value", $n_1 = n_2 = T_1 = R_1 = R_2 = R_3 = 0$ will do)
For i=1,...,n
  For j=1,...,m
    For k=1,...,p
      Set newN1=i, newN2=j, newT1=k
      Compute newR3= "the minimum R_3 that satisfies constraints C1, C3 and C4"
      Compute newR1=R_1 and newR2=R_2 where (R_1,R_2) is a solution to the equations $\frac{\partial TC}{\partial R_1} = 0$ and $\frac{\partial TC}{\partial R_2} = 0$
      Compute newTC = TC(newN1, newN2, newT1, newR1, newR2, newR3)
      If newTC <= OptimalTC then
        Set n1= newN1; n2 = newN2; T1= newT1;
        R_1= newR1; R_2= newR2; R_3= newR3;
        OptimalTC= newTC
    End
  End
End
Compute $T_2 = n_1 T_1$ and $T_3 = n_1 n_2 T_1$
Outputs: optimal values n_1, n_2, T_1, T_2, T_3, R_1, R_2, R_3, OptimalTC

The formula used in solving newR1 and newR2 are obtained using MAPLE 7 (a math solver software). The algorithm was coded to create a software using C♯ as the programming language. The screen shots of the software's user interface are shown in Figure 2 and Figure 3.

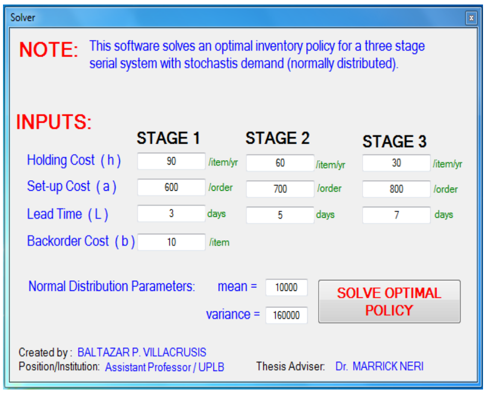

Figure 2: User interface of the software when asking for inputs

4. VALIDATION OF THE MODEL AND THE ALGORITHM

Some numerical results are obtained using the created software in C♯. Different inputs are shown in Table 1 and the corresponding optimal solutions are shown in Table 2. All examples use $L_1 = 3$ *days*, $L_2 = 5$ *days* and $L_3 = 7$ *days*.

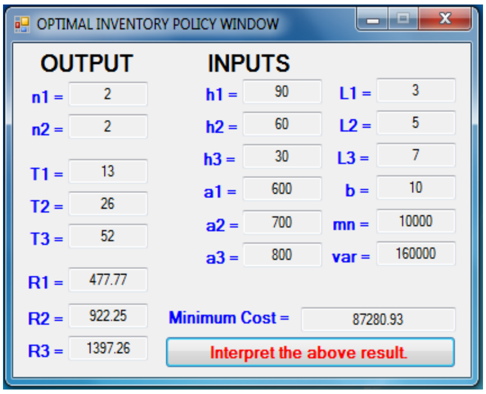

Figure 3: User interface of the software when displaying output and inputs

$T_{i's}$ are measured in days and the total cost is in pesos per year.

Table 1: Different Input Parameters for Model 1A

Ex	INPUT PARAMETERS								
	μ	σ^2	b	h_1	h_2	h_3	a_1	a_2	a_3
1	10,000	160,000	10	90	60	30	600	700	800
2	10,000	160,000	40	90	60	30	600	700	800
3	10,000	160,000	90	90	60	30	600	700	800
4	10,000	160,000	10	90	60	30	800	700	600
5	10,000	160,000	10	50	30	20	600	700	800
6	5,000	90,000	10	90	60	30	600	700	800
7	5,000	90,000	40	90	60	30	600	700	800
8	5,000	90,000	90	90	60	30	600	700	800
9	5,000	90,000	10	90	60	30	800	700	600
10	5,000	90,000	10	50	30	20	600	700	800
11	10,000	160,000	90	90	60	30	50	100	300

Ex2 to Ex5 are variations of Ex1 with some changes in the inputs. With these changes, let us compare the optimal solutions in Ex2 to Ex5 to the optimal solution in Ex1. In Ex2 and Ex3, we increased the shortage cost to Php40 and Php90. The resulting optimal R_1 increased while T_1 decreased. This is what we expect to happen since we do not want too much shortage when shortage cost is too high.

In Ex4, we change the ordering cost at stage 1 and stage 3 to 800Php and 600Php, respectively. These changes increase the review period T_1 (from 13 days to 14 days) since we want to lessen the number of order cycles per year at stage 1 due to the increase in ordering cost. In Ex5, we decrease the holding cost parameters in all stages and the resulting optimal policy tends to have an increase in review period and order-up-to-level in all stages which is, again, an expected outcome.

In Ex6 to Ex10, a different mean and variance of the demand distribution are considered but with same cost parameters as with Ex1 to Ex5, respectively. Changes in the optimal policy occur but, similar to Ex1 to Ex5, the outcomes are what we expect to happen. Ex11 shows that n_1 and n_2 can take a value other than 2 depending on the values of the cost

Table 2: Corresponding optimal policy of examples in Table 1

Ex	Optimal Policy								
	n_1	n_2	T_1	T_2	T_3	R_1	R_2	R_3	Total Cost
1	2	2	13	26	52	478	922	1,397	87,280.93
2	2	2	12	24	48	527	1,013	1,315	97,688.85
3	2	2	11	22	44	530	1,008	1,233	102,171.48
4	2	2	14	28	56	501	964	1,479	91,305.66
5	2	2	16	32	64	591	1,147	1,644	65,182.57
6	2	2	23	46	92	343	582	1,110	58,167.12
7	2	2	17	34	68	361	670	863	69,161.85
8	2	2	16	32	64	376	691	822	73,239.97
9	2	2	23	46	92	343	582	1,110	60,547.56
10	2	2	24	48	96	406	761	1,151	45,002.88
11	5	2	4	20	40	319	975	1,167	60,843.36

parameters and the mean and variance. From these results, we can conclude (at some degree) that our formulated model is valid.

Regarding the effectiveness of the algorithm in solving the model, Table 3 shows the corresponding optimal solution of Ex1 to Ex5 using GAMS. Obviously, the values in Table 2 and Table 3 are the same. Thus, the proposed algorithm is effective in solving the specific model.

Table 3: Corresponding optimal policy of Ex1 to Ex5 in Table 1 using GAMS

Ex	Optimal Policy								
	n_1	n_2	T_1	T_2	T_3	R_1	R_2	R_3	Total Cost
1	2	2	13	26	52	478	922	1,397	87,280.9
2	2	2	12	24	48	527	1,013	1,315	97,688.9
3	2	2	11	22	44	530	1,008	1,233	102,171.5
4	2	2	14	28	56	501	964	1,479	91,305.7
5	2	2	16	32	64	591	1,147	1,644	65,182.6

5. CONCLUSION

In this paper, we had formulated a general and specific model for a periodic three-stage inventory system with stochastic demand. We created a user-friendly software following the proposed algorithm. From the generated numerical results using the software, we observed that the model is valid and the algorithm is effective in solving the model. For future research, other distributions of demand can also be considered and explored in the formulation of a specific model. Other scenarios, such as products that decay over time, multiple items in the storage and limited space to store multiple items, can be integrated in the formulation of the general model. A general case (i.e. n-stage inventory policy) can also be considered for future researches.

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