Optimal Allocation of Investment to Maximize an Insurer's Prospect Value Under Risk with Exponential Claims

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ABSTRACT
This study calculates the optimal allocation of the insurer's portfolio that maximizes the prospect theory value of its gain or loss. The gain or loss is relative to the insurer's current surplus. The surplus process follows a model formulated by Liu and Yang [2]. The prospect theory minimizing strategies derived in this study are compared to the ruin probability minimizing strategy of Liu and Yang. Effects of prospect theory parameters on the investment strategy are analyzed. A simulation of the surplus process showed that using smooth normalized prospect theory (SNPT) without probability weighting is the best strategy when initial surplus is zero, while using complete SNPT (i.e. probability weighting is included) yields the best results when the initial surplus is large. The strategies are compared using finite time ruin probabilities.

Keywords
Collective risk theory, Prospect value, Optimal allocation, Exponential claims

1. INTRODUCTION
Prospect theory is a theory in behavioral economics introduced by Kahneman and Tversky [4] which serves as an alternative to the more common expected utility theory. Prospect theory captures some behaviors of a decision-maker under risk that violate the axioms of the expected utility theory such as the tendency to perceive theoretically equivalent choices differently based on how they are presented, as well as the tendency to put more weight on losses than gains in decision-making.

Meanwhile, risk theory is considered as an important field of study in actuarial science [8]. One of the foundations of this theory is the Cramér-Lundberg model, which describes the change in surplus of an insurance company facing two opposing cash flows: the incoming premiums and the outgoing claims. In this model, the company is said to have reached ruin when its surplus falls on or below zero.

Since Kahneman and Tversky instroduced the concept, many studies during the recent years have dealt with prospect theory-based portfolio selections [1, 9, 11]. The most common scenario analyzed in these studies is a one-period model with a risk-free and risky asset, wherein different portfolios are offered to the prospective investor with different distributions of asset returns. However, all of these studies dealt with an individual investor. There is no widely-known literature that applied prospect theory in the surplus process setting, where the investor is an insurance firm, hence the interest of this study.

In this study, we evaluate the value of the surplus of an insurance company, hereafter will be called *"the insurer"*, when its investment on a risky asset (a stock) and a risk-free asset (a bond) are considered, given that the insurer behaves under prospect theory. Determining the prospect theory value of a lottery undergoes two stages: the editing phase and the evaluation phase. In the editing phase, the decision-maker sets a reference point so that the outcomes that are below it are treated as losses while the outcomes above it are treated as gains. Then, in the evaluation phase, a value function of the outcomes similar to those used in the expected utility theory is used while their corresponding probabilities are weighted using a probability distortion function. The current surplus is set as the reference point and the set of differences between the final surplus and the current surplus are the outcomes.

The goal of the study is to test the adequacy of prospect theory as a framework in determining a best investment strategy for an insurer in a surplus process setting, when compared to a more objective framework. The following assumptions for the model will be used: (1) the insurer invests in a risky asset (stock) and in a risk-free asset (bond), (2) the insurer evaluates its outcomes in terms of the change in surplus, not the final surplus, (3) the insurer evaluates its surplus every time period and adjust its investments accordingly, (4) no transaction costs or tax is involved in trading. The main objective of the study is to determine an optimal allocation of investment in a risky and risk-free asset to maximize the insurer's prospect theory value under risk.

2. THEORETICAL FRAMEWORK

2.1 The Surplus Process with Investment

We set up the model of the surplus process formulated by [2] as an extension of the model by [3], which incorporates investment on a stock on the classical surplus process, by adding an investment in a risk-free asset in the form of a bond. This model follows three assumptions: (1) continuous trading is allowed, (2) no transaction cost or tax is involved in trading, and (3) assets are infinitely divisible.

The price of the bond at time t, denoted by $B(t)$ is formulated as

$$\mathrm{d}B(t) = r_0 B(t)\mathrm{d}t \tag{1}$$

where r_0 is the constant, non-negative risk-free rate. The change in the price of the stock at time t, $P(t)$, follows a geometric Brownian motion

$$\mathrm{d}P(t) = \eta P(t)\mathrm{d}t + \sigma P(t)\mathrm{d}W_t \tag{2}$$

where $\eta(\geq 0)$ is the expected instantaneous rate of return on the stock, $\sigma(>0)$ is the volatility of the stock, and $\{W_t : t \geq 0\}$ is a Wiener process.

Adding these two cash flows to the classic risk process, yielding

$$\mathrm{d}U(t) = A(t)\frac{\mathrm{d}P(t)}{P(t)} + (U(t) - A(t))\frac{\mathrm{d}B(t)}{B(t)} + \mathrm{d}R(t) \tag{3}$$

defined as the surplus process involving investment on a stock and a bond, $U(t)$, where $A(t)$ is the amount invested on the stock, and $R(t)$ is the surplus at time t according to Cramér-Lundberg model, given by

$$R(t) = u + ct - S(t) \tag{4}$$

where $c = (1+\theta)\lambda\mu$ is the constant premium rate, with $\theta(>0)$ as the relative safety loading, and $S(t) = \sum_{i=1}^{N_t} X_i$, with N_t being a Poisson process with intensity λ and X_i being an identically and independently distributed sequence of claim sizes with distribution F and mean μ.

Substituting equations (1), (2), and (4) to expressions in equation (3) and simplifying results to the equation

$$\begin{aligned}\mathrm{d}U(t) =& ((\eta - r_0)A(t) + r_0 U(t) + c)\mathrm{d}t \\ &+ \sigma A(t)\mathrm{d}W_t - \mathrm{d}S(t)\end{aligned} \tag{5}$$

with $U(0) = u$

A Hamilton-Jacobi Bellman (HJB) equation was then formulated with the survival probability, denoted by $\phi(u)$, as the objective function to be maximized and A as the control value to be modified. The HJB equation is given by:

$$\begin{aligned}\max_A \Big\{&\lambda \mathrm{E}[\phi(u - X) - \phi(u)] + \frac{1}{2}\phi''(u)\sigma^2 A(u)^2 \\ &+ \phi'(u)[c + (\eta - r_0)A(u) + r_0 u]\Big\} = 0.\end{aligned} \tag{6}$$

Note that A has become a function of the initial surplus u, instead of t.

The solution to the HJB equation yields the formula

$$A^*(u) = -\frac{\eta - r_0}{\sigma^2} \cdot \frac{\phi'(u)}{\phi''(u)} \tag{7}$$

for the ruin probability minimizing investment on the stock, given an initial surplus u.

The results were used to investigate the optimal investment strategy for exponential, gamma, and Pareto claim size distributions as well as the effects of underlying factors and parameters. For exponential claims with an average claim size of 1 and an average number of claims of 3, the result is presented in Figure 1. This

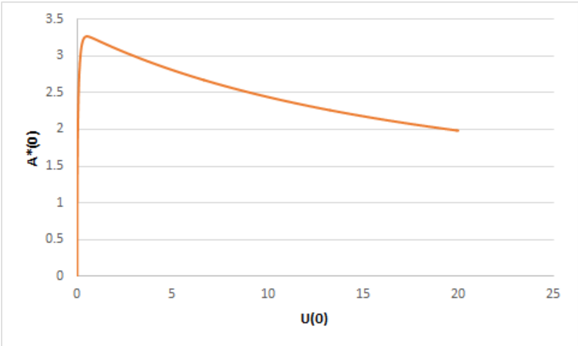

Figure 1: Ruin probability minimizing investment $A^*(U(0))$ for initial surplus $U(0) \in [0, 20]$

result showed that when the initial surplus is small, the insurer is more willing to invest in the risky asset (and even borrow at the risk-free rate) to cover the risk of claims. However, as the initial surplus increases, the insurer becomes more confident of surviving the risk of claims, so it opts for a more conservative investment strategy and invests less in the risky asset.

Further results showed that the optimal allocation in the risky asset is positively related to average number of claims while it is negatively related to the risk-free rate, the volatility of the risky asset, and the safety loading.

2.2 The Surplus Gain Random Variable

Let $X(\geq 0)$ be the exponential claim amount random variable with parameter μ and a probability density function given by

$$f_X(x) = \mu e^{-\mu x}. \tag{8}$$

and let N be the number of claims random variable, which is Poisson distributed with parameter λ, and with probability density function given by

$$\Pr[N = n] = \frac{e^{-\lambda}\lambda^n}{n!}. \tag{9}$$

Then, S is the total claim amount random variable, given by the following probability density function:

$$f_S(s) = \begin{cases} \sum_{n=1}^{\infty} \frac{\mu^n s^{n-1} e^{-\mu s}}{(n-1)!} \cdot \frac{e^{-\lambda}\lambda^n}{n!} & \text{if } s > 0 \\ e^{-\lambda} & \text{if } s = 0 \end{cases}$$

To prove this, we first note that the density of the total claim amount in a collective risk model is given by the following convolution formula:

$$f_S(s) = \sum_{n=0}^{\infty} f_X^{*(n)}(s) \cdot \Pr[N = n] \tag{10}$$

where $f_X^{*(n)}(s)$ is the density of the n-fold convolution of X [8].

First, we solve for $f_X^{*(n)}(s)$. We first note that

$$f_X^{*(0)}(s) = \begin{cases} 0 & \text{if } s > 0 \\ 1 & \text{if } s = 0 \end{cases}$$

Then, if N is Poisson distributed with mean λ,

$$\begin{aligned} f_S(0) &= f_X^{*(0)}(0) \cdot \Pr[N = 0] \\ &= 1 \cdot \frac{e^{-\lambda}\lambda^0}{0!} \\ &= e^{-\lambda}. \end{aligned} \tag{11}$$

For $s > 0$, we need to prove the expression for $f_X^{*(n)}$ for every positive integer n using mathematical induction. If $n = 1$, the density is

$$f_X(x) = \mu e^{-\mu x}, \tag{12}$$

which is the density function for an exponentially distributed random variable with parameter μ. We note that since there is only one claim, then $X = S$. So the above density function is equivalent to

$$f_X^{*(1)}(s) = \mu e^{-\mu s} \tag{13}$$

Next, assume that for $S_n = \sum_{i=1}^{n} X_i$, where the X_i's are independent and identically distributed exponential random variables,

$$f_X^{*(n)}(s) = \frac{\mu^n s^{n-1} e^{-\mu s}}{(n-1)!} \tag{14}$$

as the density function of $s(>0)$. Note that this is the density function for a gamma distribution with parameters n and μ.

Next, we solve for the density of $S_{n+1} = \sum_{i=1}^{n+1} X_i = S_n + X_{n+1} = S_n + X$. Using the convolution formula for two independent random variables, we have

$$\begin{aligned} f_X^{*(n+1)}(s) &= \int_0^s f_X(s-y) f_X^{*(n)}(y) \mathrm{d}y \\ &= \int_0^s \mu e^{-\mu(s-y)} \cdot \frac{\mu^n y^{n-1} e^{-\mu y}}{(n-1)!} \mathrm{d}y \\ &= \mu^{n+1} e^{-\mu s} \int_0^s \frac{y^{n-1}}{(n-1)!} \mathrm{d}y \\ &= \mu^{n+1} e^{-\mu s} \frac{s^n}{n(n-1)!} \\ &= \frac{\mu^{n+1} s^n e^{-\mu s}}{n!} \\ &= \frac{\mu^{n+1} s^{(n+1)-1} e^{-\mu s}}{((n+1)-1)!} \end{aligned} \tag{15}$$

Thus, we have proven that $f_X^{*(n)}(s) = \frac{\mu^n s^{n-1} e^{-\mu s}}{(n-1)!}$ for any positive integer n and finally, using (10), we have, for $s > 0$,

$$f_S(s) = \sum_{n=1}^{\infty} \frac{\mu^n s^{n-1} e^{-\mu s}}{(n-1)!} \cdot \frac{e^{-\lambda}\lambda^n}{n!} \tag{16}$$

Next, we let $Q = M - S$ where $M \sim \mathrm{N}(0, \sigma^2)$ with probability density funciton given by

$$f_M(m) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{m^2}{2\sigma^2}} \text{ for } m \in \mathbb{R} \tag{17}$$

and S is compound Poisson with exponential claims. We assume that M and S are independent. Then, $S = M - Q$ and the probability density funciton of Q can be written as

$$f_Q(q) = \int_{-\infty}^{\infty} f_S(m-q) f_M(m) \mathrm{d}m. \tag{18}$$

Substituting (16) and (17) to (18), and after some simplification, we have

$$f_Q(q) = \frac{e^{\mu q - \lambda}}{\sqrt{2\pi\sigma^2}} \int_q^{\infty} e^{-\left(\mu m + \frac{m^2}{2\sigma^2}\right)} \vartheta(m) \mathrm{d}m. \tag{19}$$

where

$$\vartheta(m) = \sum_{n=1}^{\infty} \frac{(\mu\lambda)^n (m-q)^{n-1}}{n!(n-1)!} \tag{20}$$

In order to simplify the evaluation of the prospect theory value, we simplify the surplus process by discretizing (5), yielding

$$\begin{aligned} \Delta U_t =& (\eta - r_0) A_t \Delta t + r_0 U_t \Delta t + \sigma A_t \Delta W_t \\ &+ c\Delta t - \Delta S_t. \end{aligned} \tag{21}$$

Since we are only interested in evaluating the prospect theory value per time period, (21) simplifies to

$$\begin{aligned} U_1 - U_0 =& (\eta - r_0) A_0 + r_0 U_0 + \sigma A_0 (W_1 - W_0) \\ &+ c - \sum_{i=1}^{N} X_i. \end{aligned} \tag{22}$$

For ΔS_t, we used the fact that ΔN_t, following a compound Poisson process with intensity λ, has an identical distribution to N as it is defined in the beginning of Section 2.2.

Now, let H be the gain random variable for the surplus process. From (22), we have

$$\begin{aligned} H &= U_1 - U_0 \\ &= (\eta - r_0)A_0 + r_0 U_0 + \sigma A_0 Z + c - S \\ &= \sigma A_0 Z - S + k \end{aligned} \tag{23}$$

where S is the total claim amount random variable, $Z \sim \mathrm{N}(0,1)$, and k is a constant equal to $(\eta - r_0)A_0 + r_0 U_0 + c$. Now, let $M = \sigma A_0 Z$. Then,

$$H = M - S + k \tag{24}$$

To derive the density of H using the following procedure, we first calculate $\Pr[H \leq h]$.

$$\begin{aligned} \Pr[H \leq h] &= \Pr[M - S + k \leq h] \\ &= \Pr[M - S \leq h - k] \end{aligned} \tag{25}$$

From the property of the normal distribution, we conclude that $M \sim \mathrm{N}(0, \sigma^2 A_0^2)$. Hence,

$$\Pr[H \leq h] = \Pr[Q \leq h - k] \tag{26}$$

Now, using (18), we have

$$F_Q(h-k) = \int_{-\infty}^{h-k} f_Q(q)\mathrm{d}q \tag{27}$$

where

$$f_Q(q) = \frac{e^{\mu q - \lambda}}{\sqrt{2\pi\sigma^2 A_0^2}} \int_q^{\infty} e^{-\left(\mu m + \frac{m^2}{2\sigma^2 A_0^2}\right)} \vartheta(m)\mathrm{d}m \tag{28}$$

and

$$\vartheta(m) = \sum_{n=1}^{\infty} \frac{(\mu\lambda)^n (m-q)^{n-1}}{n!(n-1)!}. \tag{29}$$

Now, we use a change of variables. Let $r = q + k$. Then, the integral becomes

$$F_H(h) = \int_{-\infty}^{h} f_H(r)\mathrm{d}r. \tag{30}$$

And thus, the density of H is given by

$$f_H(h) = \frac{e^{\mu(h-k) - \lambda}}{\sqrt{2\pi\sigma^2 A_0^2}} \int_{h-k}^{\infty} e^{-\left(\mu m + \frac{m^2}{2\sigma^2 A_0^2}\right)} \varrho(m)\mathrm{d}m \tag{31}$$

where

$$\varrho(m) = \sum_{n=1}^{\infty} \frac{(\mu\lambda)^n (m-h+k)^{n-1}}{n!(n-1)!}. \tag{32}$$

However, the density given by (31) only applies when $A_0 > 0$. For the trivial case $A_0 = 0$, the density is simply given by

$$f_H(h) = e^{\mu(h-\ell)-\lambda} \sum_{n=1}^{\infty} \frac{(\mu\lambda)^n (-h+\ell)^{n-1}}{n!(n-1)!} \tag{33}$$

where $\ell = r_0 U_0 + c$.

2.3 Prospect Theory Value

For a given lottery with a set of finite outcomes $X = (x_1, x_2, x_3, ..., x_n)$ and a set of corresponding probabilities $P = (p_1, p_2, p_3, ..., p_n)$, the prospect theory value is given by

$$PT(X) = \sum_{i=1}^{n} v(x_i) w(p_i) \tag{34}$$

where $v(x_i)$ is the value function similar to the utility function in expected utility theory and $w(p_i)$ is the probability weighting (or probability distortion) function. Note that the set of outcomes X are presented as gains and losses, not as final wealth.

The value function to be used in this study is the one proposed by [4], defined as

$$v(x) = \begin{cases} x^{\alpha_1} & \text{if } x \geq 0 \\ -\beta|x|^{\alpha_2} & \text{if } x < 0 \end{cases}$$

where β is the loss sensitivity parameter and α_1 and α_2 are the parameters that set the curvature of the S-shaped curve of the value function. In testing a best fit model of the value function to empirical data, [10] obtained the following estimates for the parameters: $\alpha_1 = 0.39$, $\alpha_2 = 0.69$, and $\beta = 2.02$

Meanwhile, the probability weighting function to be used is the one introduced by [5] is

$$w(p) = \exp(-(-\ln(p))^{\gamma}) \tag{35}$$

where γ is the weighting parameter. This probability weighting function captures the risk-seeking attitude for large probabilities of losses and risk-averse attitude for large probabilities of gains. [10] estimated γ to be 0.44.

However, the prospect theory value defined previously can only be applied in a discrete set of finite outcomes. In this study, our outcome is defined by the random variable H, which has a continuous probability distribution. Hence, a variant of prospect theory for continuous outcomes will be used, called *smooth normalized prospect theory*.

Let p be a probability measure, v and w be the value and probability weighting functions, respectively, and ε is a parameter wherein outcomes that differ by ε are treated as the same. Then the smooth normalized prospect theory value of X is given by

$$SNPT(X) = \frac{\displaystyle\int_{\Omega} v(x) w\left(\int_{x-\varepsilon}^{x+\varepsilon} \mathrm{d}p\right)\mathrm{d}x}{\displaystyle\int_{\Omega} w\left(\int_{x-\varepsilon}^{x+\varepsilon} \mathrm{d}p\right)\mathrm{d}x} \tag{36}$$

This equation is formulated from a result in [7] to get the prospect theory value of outcomes with continuous probability distributions.

For the gain random variable H, the smooth normalized prospect theory value is given by

$$SNPT(H) = \frac{\int_{-\infty}^{\infty} v(h) w\big(\int_{h-\varepsilon}^{h+\varepsilon} f_H(r)\mathrm{d}r\big)\mathrm{d}h}{\int_{-\infty}^{\infty} w\big(\int_{h-\varepsilon}^{h+\varepsilon} f_H(r)\mathrm{d}r\big)\mathrm{d}h} \quad (37)$$

For $A_0 = 0$, the upper bounds of the outer integrals become ℓ.

2.4 Approximating the Integral

Trapezoidal rule is employed in calculating all integrals. An integral represented by an area under a curve $y = f(x)$ over $[a, b]$ can be approximated by a series of trapezoids that lie above the sub-intervals $[x_k, x_{k+1}]$. The combined area of the trapezoids is given by:

$$T(f, h) = \frac{h}{2} \sum_{k=1}^{M} (f(x_{k-1}) + f(x_k)) \quad (38)$$

where h is the length of each sub-interval and M is the number of sub-intervals [6]. To solve numerically for $SNPT(H)$, the intrinsic function *inttrap* in Scilab was used in implementing the trapezoidal rule for all integrals involved. However, for numerical integration to be feasible, the bounds of the integrals needed to be adjusted.

For $f_H(h)$, we choose $\bar{m}$ such that $\Pr[M > \bar{m}] \leq 10^{-4}$. The tolerance level was chosen to truncate the tail of the distribution for efficiency in using the trapezoidal rule. Furthermore, $\bar{n}$ was also chosen such that $\Pr[N > \bar{n}] \leq 10^{-4}$. Thus, (31) becomes

$$f_H(h) = \frac{e^{\mu(h-k)-\lambda}}{\sqrt{2\pi\sigma^2 A_0^2}} \int_{h-k}^{\bar{m}} e^{-\left(\mu m + \frac{m^2}{2\sigma^2 A_0^2}\right)} \varrho(m)\mathrm{d}m. \quad (39)$$

The same limiting bound $\bar{n}$ is applied to (33).

We set $\bar{s}$ such that $\Pr[S > \bar{s}] \leq 10^{-4}$, where $f_S(s) = \sum_{n=1}^{\bar{n}} \frac{\mu^n s^{n-1} e^{-\mu s}}{(n-1)!} \cdot \frac{e^{-\lambda}\lambda^n}{n!}$, for $s > 0$. Then, (37) becomes

$$SNPT(H) = \frac{\int_{k-(\bar{m}+\bar{s})}^{k+\bar{m}} v(h) w\left(\int_{h-\varepsilon}^{h+\varepsilon} f_H(r)\mathrm{d}r\right)\mathrm{d}h}{\int_{k-(\bar{m}+\bar{s})}^{k+\bar{m}} w\left(\int_{h-\varepsilon}^{h+\varepsilon} f_H(r)\mathrm{d}r\right)\mathrm{d}h} \quad (40)$$

For $A_0 = 0$, the upper and lower bounds of the outer integrals become ℓ and $\ell - \bar{s}$, respectively.

Microsoft Excel is used to obtain $\bar{n}$, $\bar{s}$, and $\bar{m}$. Note that $\bar{m}$ is unique for every A_0.

3. RESULTS AND DISCUSSION

3.1 The Distribution of H

Using a Scilab program, the distribution of H for $A_0 = 0, 0.5, 1, 1.5, ..., 19.5, 20$ and $U_0 = 0, 1, 2, 3, ..., 19, 20$ is determined. The case $A_0 > U_0$ is considered to be the case where the insurer can borrow $A_0 - U_0$ at the risk-free rate. The parameters used are $\eta = 0.1$, $r_0 = 0.01$, $\sigma = 0.3$, $\theta = 0.2$, $\lambda = 3$ and $\mu = 1$, the same parameters used in [2]. The resulting $\bar{n}$, $\bar{s}$, are 11 and 21, respectively. $\bar{m}$ is given by a vector of values corresponding to each A_0. For each integral, trapezoidal rule is used with 25 subintervals. The results are shown in Figure 2 and Figure 3.

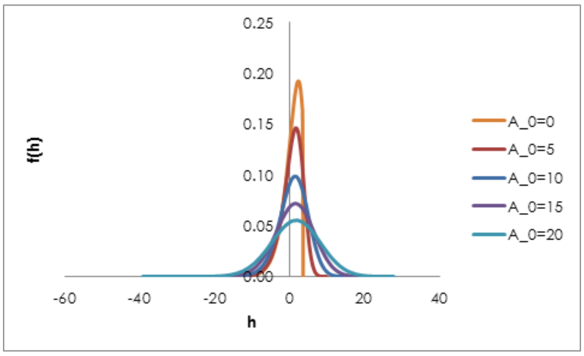

Figure 2: The distribution of H for $U_0 = 0$

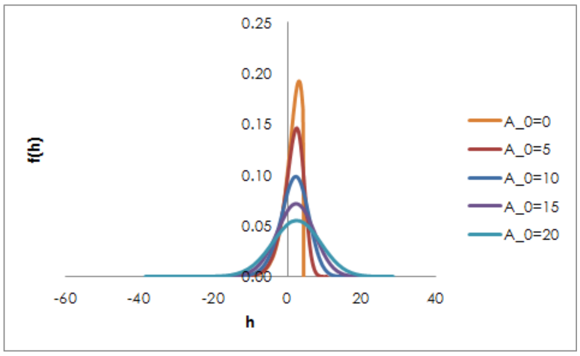

Figure 3: The distribution of H for $U_0 = 20$

From the two figures, note that the resulting distributions for $U_0 = 0$ and $U_0 = 20$ have near-identical shapes, which is in fact, the case for all $U_0 = 0, 1, 2, ..., 20$. This means that the initial surplus has minimal effect on the distribution of the change in the surplus. However, the allocation in the risky asset heavily affects the distribution, since it affects the variance of the random variable M. It is important to note that the peak of the distribution always lies above $h > 0$, which means a greater probability of having gains than losses but the tail of the distribution is thicker on the left side of the origin, which means that the it is more possible to render very heavy losses than very large gains, albeit all extreme events have very low probability. It can also be observed from the two figures that as A_0 increases, the distribution becomes more platykurtic and the tails become heavier, which increases the probability of incurring a loss.

3.2 Complete SNPT Behavior

We also compute $SNPT(H)$ by employing the trapezoidal rule in Scilab. The prospect theory parameters used are the ones estimated by [10], namely: $\alpha_1 = 0.39$, $\alpha_2 = 0.69$, $\beta = 2.02$, and $\gamma = 0.44$. Meanwhile, we set $\epsilon = 0.1$. The result is shown in Figure 4.

It can be observed that as the allocation on the risky asset A_0 increases, the smooth normalized prospect

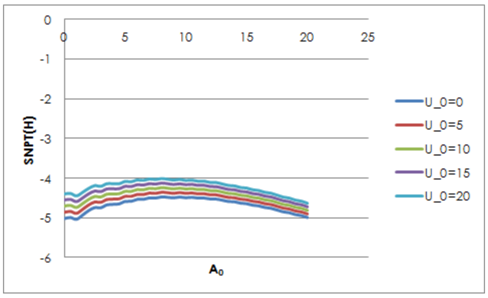

Figure 4: Smooth normalized prospect theory value of H with $\alpha_1 = 0.39$, $\alpha_2 = 0.69$, $\beta = 2.02$, and $\gamma = 0.44$.

theory value $SNPT(H)$ roughly increases until it reaches $A_0 = 8$ and decrease thereafter. Meanwhile, as the initial surplus U_0 increases, the graph merely shifts upward but retains its shape. This means that the complete SNPT insurer is willing to invest an amount of 8 in the risky asset, regardless of initial wealth, since the distributions of H are nearly the same for all values of U_0 considered. It is also important to note that the insurer is willing to risk a considerable amount in its investment since it is already expecting to lose money, signifying pessimistic behavior, but it will not take risks when the probability of losses clearly outweighs the chances of gains.

3.3 No Loss Sensitivity

If we set $\beta = 1$, then the insurer is not sensitive to its losses. This results to an increase in the prospect value of the surplus as shown in Figure 5. Furthermore, the peak of the graphs shifted to $A_0 = 12.5$. This means that an insurer that is not loss sensitive is willing to invest more in the risky asset and is more optimistic with its outcomes, which is logically consistent.

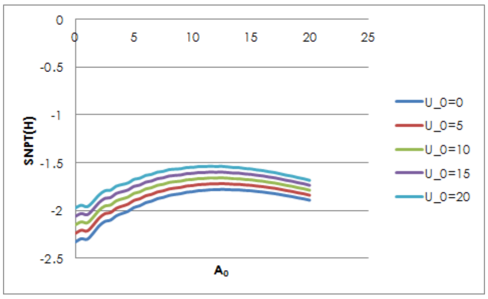

Figure 5: Smooth normalized prospect theory value of H with $\alpha_1 = 0.39$, $\alpha_2 = 0.69$, $\beta = 1$, and $\gamma = 0.44$.

3.4 No Curvature in the Value Function

If we set both α_1 and α_2 to be 1, then the value function is linear. Figure 6 shows that this results to a decrease in the overall prospect value of the surplus but an increase in the investment in the risky asset. This is because the marginal value of the gain and the loss are no longer diminishing, this makes the insurer expect to lose more, making it more willing to invest in the risky asset.

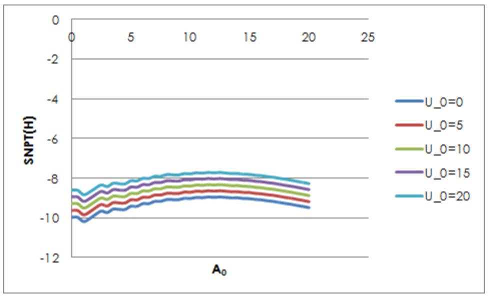

Figure 6: Smooth normalized prospect theory value of H with $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta = 2.02$, and $\gamma = 0.44$.

3.5 No Probability Weighting

To model the surplus's prospect theory value without weighting the probabilities, simply setting γ to be equal to 1 is not enough. We also need to set ϵ to be 0 to remove the effect of weighting similar outcomes. Thus, the results are shown in Figure 7.

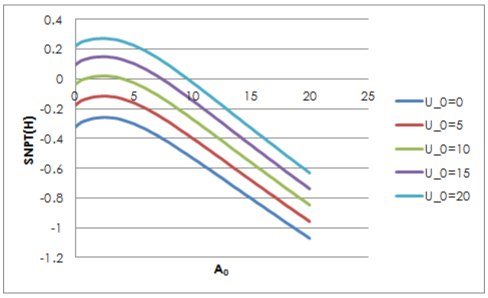

Figure 7: Smooth normalized prospect theory value of H with $\alpha_1 = 0.39$, $\alpha_2 = 0.69$, $\beta = 2.02$, and $\gamma = 1$.

The figure shows that if the insurer does not overweight its probabilities, it becomes more conservative in its strategy since it does not seek the chance of gains that much. However, the overall prospect value increases since the insurer is also not overweighting very small probabilities of large losses. The insurer is expecting a gain but a small loss is enough to keep him from risking more than an amount of 2.5 in the risky asset. This result is the one closest to the result in [2].

3.6 Absence of Prospect Theory Behavior

If we set $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta = 1$, and $\gamma = 1$, then the insurer will evaluate its gain or loss in terms of expected value. Figure 8 shows that investment in the risky asset increases as the average gain in the surplus increases. This makes the insurer invest the maximum amount it has in the risky asset, signifying risk-seeking behavior. This strategy is not recommended in practice.

3.7 Comparison of Optimal Investment Strategies

Let us denote the investment strategies of an insurer minimizing its probability of ruin as discussed in Section 2.1, an insurer engaging in complete SNPT behavior

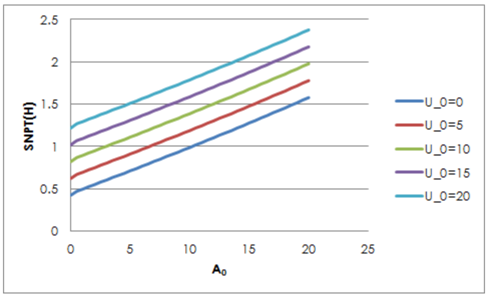

Figure 8: Smooth normalized prospect theory value of H with $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta = 1$, and $\gamma = 1$.

as discussed in Section 3.2, and an insurer engaging in SNPT behavior without the probability weighting as discussed in Section 3.5 as strategy A, B, and C, respectively. Figure 9 shows the graphs of the three strategies.

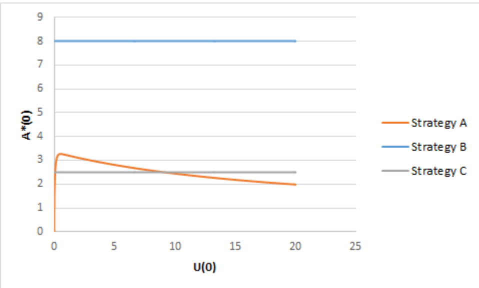

Figure 9: Optimal allocation of investment in the risky asset for initial surplus 0 to 20.

The figure shows that in this setup, the behavior of an SNPT insurer minus the probability weighting opts for an investment strategy almost identical to that of a ruin probability minimizing insurer, while an insurer engaging in complete SNPT behavior would employ a less conservative, more risk-seeking strategy.

A simulation of the surplus process for 10 time periods is run using Microsoft Excel. Insurers A, B, and C each start with initial surplus $= u$, are faced with the total claim amounts and the movement of the stock price random varible, and respond with their corresponding investment strategies for every time period. Each random sample is simulated by first generating a uniform normal random number p using the *RAND* function, then calculating the approximate inverse cumulative distribution function of p with respect to the intended distribution of the sample. A realization of the process is shown in Table 1.

25,000 paths are simulated and the probability of ruin before time 10, denoted by $\psi(u, 10)$ is approximated for each investment strategy. A finite time ruin probability is considered since in practice, finite time ruin probabilities are more important than absolute ruin probabilities, as intervening factors that can drastically affect the model like economic recession and regime switching in the short run render long run predictions obsolete. The results for $u = 0, 5, 10$ are shown in Table 2.

t	$U_t(A)$	$U_t(B)$	$U_t(C)$
0	0	0	0
1	1.94	1.5781	1.8269
2	4.2973	3.8264	4.1916
3	8.0991	9.6974	7.8374
4	10.2018	9.4705	9.9594
5	6.8516	6.91987	6.6098
6	9.5387	9.2941	9.2965
7	11.3586	11.0334	11.1064
8	9.0457	7.1641	8.74602
9	3.1817	-0.7423	2.8712
10	7.4893	—	6.8749

Table 1: A sample path of the surplus process using strategies A, B, and C for $u = 0$.

u	$\psi_A(u, 10)$	$\psi_B(u, 10)$	$\psi_C(u, 10)$
0	0.35732	0.41488	0.35224
5	0.12464	0.13388	0.1204
10	0.07584	0.0722	0.0738

Table 2: The approximated probabilities of ruin before time 10 using 25,000 simulations.

For initial surplus 0, Strategy C yielded the least average finite time ruin probability, followed by Strategy A, and then Strategy B. Thus, for a time frame of 10 periods, it can be concluded that the SNPT insurer without probability weighting has the best investment strategy, only better than the ruin probability minimizing insurer by a small margin. This may be a result of the formulation of Strategy A, wherein the increasing allocation of investment in the risky asset from zero up to the peak at small values of initial surplus is considered as a poor strategy while facing the risk of claims, while in comparison, the SNPT insurer without probability weighting invests 2.5 in the risky asset even at zero initial surplus. Meanwhile, the complete SNPT insurer has the poorest strategy among the three, which implies that investing very large amounts in the risky asset, as that in an expected value approach, would more likely result to ruin when the initial surplus is at the lowest since it is where the risk of ruin is largest.

At initial surplus 10, Strategy B yielded the least average ruin probability, followed by Strategy C, and then Strategy A, which mean that when the initial surplus is large enough for the surplus to drift away from zero, the risk of ruin becomes small that taking risks in investment can be considered as a reliable strategy. This is particularly the case in this setting, where the claims follow a light-tailed distribution. However, the increase in the initial surplus also resulted to the difference among the finite time ruin probabilities of the three strategies to be less significant. When the initial surplus is large enough, any strategy will give almost the same finite time ruin probability.

4. CONCLUSION

Prospect theory is a widely recognized theory used to model the behavior of a decision-maker which is consistent with how people think in reality. A decision-maker with prospect theory behavior thinks only in terms of gains and losses, is sensitive to losses, and tends to overweight small probabilities and underweight very large probabilities. However, it is only a descriptive model and thus, is not highly recommended for modelling objective behavior like optimal strategies.

For an insurer with prospect theory behavior that faces exponential claims, it is found out that the optimal allocation of investment in the risky asset is consistent for any initial surplus since the distribution of the gain in surplus for any initial surplus is near identical to one another. An insurer with prospect theory behavior is also found out to be willing to invest (or even borrow to invest) a considerable amount in stocks if it is expecting only a loss while it is willing to risk only a small amount if it expects to gain or to be able to cover the claims. It was also shown that an SNPT insurer that does not distort its probabilities chooses an investment strategy close to that of the ruin probability minimizing insurerof Liu and Yang [2]. This is further supported by the results of the simulation for finite time ruin probabilities. Moreover, the simulation showed that a prospect theory model without probability weighting yields a better finite time ruin probability than a ruin probability minimizing model. A complete prospect theory model also provides a better model than the two when the initial surplus is large enough but with less significant differences. This implies that a prospect theory framework can be adequate not only as a descriptive tool but also to challenge objective optimal decision models. At best, it provides better models than the ruin probability minimizing model of [2] in terms of short run finite time ruin, which is a more useful time frame to consider than long run time, especially when dealing with immediate changes in investment strategies.

Mathematical modelling of "irrational" behavior can be applied to many other areas in the insurance business, such as analayzing the behavior of policyholders, agency problems, response to impending natural or man-made disasters, and business competition. We hope that with this study, interest in studying the application of prospect theory, and behavioral economics in general, would flourish.

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