Analyses on a modified inexact primal-dual method for removing multiplicative noise

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ABSTRACT
Multiplicative noise that commonly appears in digital medical imaging is removed using a total variation model presented in the article of Del Rosario and Neri (2017). We present here details on the theoretical analyses on the derivation of the model and its solution.

Keywords: Multiplicative noise, variation model, primal-dual method

1. INTRODUCTION
In digital image restoration, variational models have proven to be effective in reconstructing images corrupted with noise. In the seminal paper of Rudin, Osher, and Fatemi [10], additive Gaussian noise was effectively removed using a total variation model with L^2 fidelity term. Their model is

$$\min_{u \in \mathrm{BV}(\Omega)} \frac{1}{2}\int_{\Omega}(u-d)^2 + \alpha \int_{\Omega} |\nabla u| \tag{1}$$

where d is the observed image, Ω is a closed rectangular region with Lipschitz continuous boundary, BV denotes the space of functions of bounded variations, and ∇ is the difference operator. Since then, numerous research papers have improved on the ROF model to handle even simultaneous noise removal and blur removal. For outlier noise or impulse noise, Nikolova [9] has shown that the variational model with L^1 norm on the fidelity term is more suitable for filtering.

Compared to additive noise removal, only a few variational approaches have been proposed that focus in handling multiplicative noise. This is a more complex noise model commonly appearing in various imaging applications such as laser images, microscopic images, ultrasound images, synthetic aperture radar (SAR) images, etc. One of the first is a nonconvex total variation model with a fitting term derived from a maximum *a posteriori* estimator introduced in [2] by Aubert and Aujol (AA). They implemented a gradient-descent method to solve their model. Dong and Zeng [6] extended the AA model by adding a quadratic penalty term to ensure convexity. A primal-dual method proposed in [4] was applied to solve their model.

In [5], a modified version of Dong and Zeng's convex variational model for the multiplictive noise removal problem was presented. This paper is an expanded version of that paper, and herein we show proofs of theorems given in [5].

2. NONCONVEX MODEL
In the formulation of the model, the assumption is that the original image u has been corrupted by some multiplicative noise η that follows a gamma distribution with mean equal to one. Given the multiplicative noise model $f = u\eta$, we consider f, u, and η as instances of random variables $\mathscr{F}, \mathscr{U}$ and $\mathscr{N}$.

It is assumed that $\mathscr{U}$ follows a Gibbs prior i.e.

$$P_{\mathscr{U}}(u) = \frac{1}{Z}e^{-\gamma\varphi(u)} \tag{2}$$

where Z is a normalizing constant, γ is a free parameter and φ is a non-negative function.

Further, $\mathscr{N}$ is an independent and identically distributed noise which follows a gamma distribution with the probability density function (pdf) given by

$$P_{\mathscr{N}}(\eta) = \frac{1}{\theta^K \Gamma(K)}\eta^{K-1}e^{-\eta/\theta}, \qquad \eta > 0 \tag{3}$$

where Γ is the usual gamma function, and θ and K denote the scale and shape parameters, respectively.

It is known that the mean of η is $K\theta$ and the variance is $K\theta^2$ [7]. Assuming the mean equals one, the variance is $\frac{1}{K}$. The pdf in (3) becomes

$$P_{\mathscr{N}}(\eta) = \frac{K^{(K)}}{\Gamma(K)}\eta^{K-1}e^{-K\eta}, \qquad \eta > 0. \tag{4}$$

From a statistical perspective, the problem of finding the most likely image u given the noisy image f, is equivalent to the problem of maximizing the probability density function (pdf) $P_{\mathscr{U}|\mathscr{F}}(u|f)$ or the MAP estimation problem given by

$$\max P_{\mathscr{U}|\mathscr{F}}(u|f) = \max P_{\mathscr{U}}(u)P_{\mathscr{F}|\mathscr{U}}(f|u). \tag{5}$$

Taking the negative log likelihood, (5) becomes the minimization problem

$$\min(-\log P_{\mathscr{U}}(u) - \log P_{\mathscr{F}|\mathscr{U}}(f|u)). \tag{6}$$

The first term in (6) can easily be computed from (2) while the second term can be solved from (4) by a standard pdf property, $P_{\mathscr{F}|\mathscr{U}}(f|u) = \left(\frac{1}{u}\right)P_{\mathscr{N}}\left(\frac{f}{u}\right)$. It follows that

$$P_{\mathscr{F}|\mathscr{U}}(f|u) = \frac{K^{(K)}}{u^K\Gamma(K)}f^{K-1}e^{\left(\frac{-Kf}{u}\right)}.$$

Hence, over the discretized image,

$$\begin{aligned}&-\log P_{\mathscr{U}}(u)\ -\ \log P_{\mathscr{F}|\mathscr{U}}(f|u)\\ &= \sum_{s\in \mathrm{S}}\left(\gamma\varphi(\mathscr{U}(s)) + K\left(\log\mathscr{U}(s) + \frac{\mathscr{F}(s)}{\mathscr{U}(s)}\right)\right)\end{aligned}$$

where S is the set of pixels in the image.

In the continuous case, this amounts to

$$\int_\Omega \varphi(u) + \frac{K}{\gamma}\int_\Omega\left(\log u + \frac{f}{u}\right)$$

where Ω denotes the image domain. Setting $\varphi(u) = |\nabla u|$, the variation model of Aubert and Aujol (AA) is obtained [2],

$$\min_{u\in S(\Omega)} E(u) = \int_\Omega |\nabla u| + \lambda\int_\Omega\left(\log u + \frac{f}{u}\right) \tag{7}$$

where $S(\Omega) = \{u \in BV(\Omega), u > 0\}$ and $f > 0 \,\text{in}\, L^\infty(\Omega) = \{f : \int_\Omega |f|^\infty < \infty\}$ [2]. The closure of S is denoted by $\bar{S}$.

A solution u^* to (7) should satisfy the Euler-Lagrange equations

$$\begin{cases} \nabla^T\frac{\nabla u}{|\nabla u|} + \lambda\left(\frac{u-f}{u^2}\right) = 0, & \text{if } |\nabla u| \neq 0\\ \nabla u = 0, & \text{otherwise}\end{cases}$$

The fitting (second) term in (7) is nonconvex. However, under sufficient conditions, the existence and uniqueness of the solution can be ensured. A gradient-descent method was implemented in [2] to generate iterative solutions to problem (7).

3. CONVEX MODEL

To overcome the nonconvexity of the AA model, Dong and Zeng [6] proposed a new convex model by adding a quadratic penalty term based on statistical properties of the multiplicative gamma noise. A random variable $Y = \frac{1}{\sqrt{\eta}}$ was considered. Given that $E(\eta) = 1$, the following properties were derived:

1. The mean values of Y and Y^2 are $E(Y) = \frac{\sqrt{K}\Gamma\left(K-\frac{1}{2}\right)}{\Gamma(K)}$ and $E(Y^2) = \frac{K}{K-1}$, respectively.
2. $\lim_{K\to+\infty} E((Y-1)^2) = 0$

Since $f = u\eta$, then $Y = \sqrt{\frac{u}{f}}$. A quadratic penalty term based on property 2 was introduced to (7) resulting in the new model given by

$$\min_{u\in\bar{S}(\Omega)} \lambda\int_\Omega |\nabla u| + \int_\Omega\left(\log u + \frac{f}{u}\right) + \alpha\int_\Omega\left(\sqrt{\frac{u}{f}} - 1\right)^2 \tag{8}$$

with a penalty parameter $\alpha > 0$. We refer to model (8) as the DZ model.

Problem (8) is strictly convex whenever $\alpha \geq \frac{2\sqrt{6}}{9}$. This convexity leads to uniqueness of the solution, and it makes (8) a suitable convex approximation of the nonconvex AA model (7). Dong and Zeng applied the primal-dual method proposed in [4] to generate iterative solutions to their model.

4. PROPOSED MODEL

We construct a modified version of Dong and Zeng's model based on the random variable $Y = \frac{1}{\eta}$. For such a variable Y, we obtain several results.

THEOREM 1. *The mean values of Y and Y^2 are $E(Y) = \frac{K}{K-1}$ and $E(Y^2) = \frac{K^2\Gamma(K-2)}{\Gamma(K)}$, respectively.*

PROOF. Recall the pdf of η given by $P_{\mathscr{N}}(\eta)$ in (3) and the assumption that the mean of η is 1, i.e., $K\theta = 1$. Since Y is a function of the random variable $\mathscr{N}$, i.e., $Y = g(\eta) = \frac{1}{\eta}$, the expectation of Y is given by

$$E(Y) = \int_{-\infty}^{+\infty} g(\eta)P_{\mathscr{N}}(\eta).$$

We then get

$$E(Y) = \int_0^{+\infty}\frac{1}{\eta}\ \ \frac{1}{\theta^K\Gamma(K)}\ \ \eta^{K-1}e^{-\eta/\theta}\ \ d\eta.$$

Based on the property of the Gamma distribution

$$\int_0^{+\infty} \frac{1}{\theta^K \Gamma(K)} \quad \eta^{K-1} e^{-\eta/\theta} \quad d\eta = 1, \;\; K > 0 \;\; \text{and} \;\; \theta > 0$$

and $\Gamma(K+1) = K\Gamma(K)$ for any $K > 0$, we obtain

$$\begin{aligned} E(Y) = \quad & \frac{\Gamma(K-1)}{\theta\Gamma(K)} \int_0^{+\infty} \frac{1}{\theta^{K-1}\Gamma(K-1)} \; \eta^{K-2} e^{-\eta/\theta} \; d\eta \\ = \quad & \frac{K}{K-1} \end{aligned}$$

when $K > 1$. Similarly,

$$\begin{aligned} E(Y^2) = \quad & \int_0^{+\infty} \frac{1}{\eta^2} \frac{1}{\theta^K \Gamma(K)} \quad \eta^{K-1} e^{-\eta/\theta} \; d\eta \\ = \quad & \frac{\Gamma(K-2)}{\theta^2 \Gamma(K)} \int_0^{+\infty} \frac{1}{\theta^{K-2}\Gamma(K-2)} \eta^{K-3} e^{-\eta/\theta} \; d\eta \\ = \quad & \frac{K^2 \Gamma(K-2)}{\Gamma(K)}. \end{aligned}$$

□

Using Theorem 1 and the definition of the gamma function, we obtain the limit result.

THEOREM 2. $\lim\limits_{K \to +\infty} E((Y-1)^2) = 0.$

PROOF. From Theorem 1, we readily have

$$\begin{aligned} E((Y-1)^2) = \quad & E(Y^2) - 2E(Y) + E(1) \\ = \quad & \frac{K^2 \Gamma(K-2)}{\Gamma(K)} - \frac{2K}{K-1} + 1. \end{aligned}$$

Based on the property of the Gamma function [1],

$$\lim_{K \to +\infty} \frac{\Gamma(K+\alpha)}{\Gamma(K) K^\alpha} = 1, \qquad \alpha \in \mathbb{R}$$

and taking $\alpha = -2$, we can obtain

$$\lim_{K \to +\infty} \frac{K^2 \Gamma(K-2)}{\Gamma(K)} - \frac{2K}{K-1} + 1 = 1 - 2 + 1 = 0.$$

□

Since $f = u\eta$, then $Y = \dfrac{1}{\eta} = \dfrac{u}{f}$. From Theorem 2, we obtain a new quadratic penalty term resulting to our proposed model given by

$$\min_{u \in \bar{S}(\Omega)} E(u) = \lambda \int |\nabla u| + \int \left(\log u + \frac{f}{u} \right) + \alpha \int \left(\frac{u}{f} - 1 \right)^2. \tag{9}$$

Although (9) is nondifferentiable because of the TV functional, the model is convex for a specific range of α, as the next theorem would show.

THEOREM 3. *If* $\alpha > \dfrac{1}{54}$, *then (9) is strictly convex.*

PROOF. With $t \in \mathbb{R}^+$ and a fixed α, we define a function g as

$$g(t) = \log t + \frac{1}{t} + \alpha (t-1)^2.$$

The first derivative of g is

$$g'(t) = t^{-1} - t^{-2} + 2\alpha(t-1)$$

and the second derivative is given by

$$g''(t) = -t^{-2} + 2t^{-3} + 2\alpha.$$

We want to show that $g''(t) \geq 0$, $t \in \mathbb{R}^+$, for some values of α. Note that the third derivative of g is

$$g^{(3)}(t) = 2t^{-3} - 6t^{-4}$$

and the root is $t = 3$ which implies that $t = 3$ is a critical point of $g''(t)$. At $t = 3$,

$$g^{(4)}(t) = -6t^{-4} + 24t^{-5} > 0$$

i.e. $t = 3$ is a minimum of $g''(t)$.

Thus, at $t = 3$, g'' reaches its unique minimum, $g''(3) = \dfrac{-1 + 54\alpha}{27}$. It follows that if $\alpha \geq \dfrac{1}{54}$, $g'' \geq 0$, i.e., g is convex. Moreover, strict convexity of g follows when $\alpha > \dfrac{1}{54}$.

By letting $t = \dfrac{u}{f}$, we deduce that the second and third terms of $E(u)$ in (9) is strictly convex. Since the TV functional is convex, it also follows that $E(u)$ in (9) is strictly convex if $\alpha > \dfrac{1}{54}$ Furthermore, based on the convexity of $\bar{S}(\Omega)$, we conclude that our proposed model (9) is strictly convex. □

This condition for convexity of our proposed model (9) is more relaxed compared to that for Dong and Zeng's model (8).

5. EXISTENCE AND UNIQUENESS OF A SOLUTION

We can conclude by Theorem 3 that with a suitable α, (9) is a convex approximization of the nonconvex AA model (7). A solution u^* to (9) should satisfy the Euler-Lagrange equations

$$\begin{cases} \lambda \;\; \nabla^T \dfrac{\nabla u}{|\nabla u|} + \dfrac{u-f}{u^2} + 2\alpha \left(\dfrac{u-f}{f^2} \right) & = 0, \quad \text{if} \;\; |\nabla u| \neq 0 \\ \hfill \nabla u & = 0, \quad \text{otherwise} \end{cases}$$

We now prove the existence and uniqueness of a solution to (9). Recall that a function f is in $L^\infty(\Omega)$ if it is essentially bounded, i.e., if $\|f\|_\infty = \max_i |f_i| < \infty$.

THEOREM 4. *Let f be in $L^\infty(\Omega)$ with $\inf_\Omega f > 0$; then the model (9) has a solution u^* in $BV(\Omega)$ satisfying*

$$0 < \inf_\Omega f \le u^* \le \sup_\Omega f.$$

PROOF. The structure of the proof closely follows that of Theorem 3.6 in [6] with the term $\sqrt{\frac{u}{f}}$ replaced by $\frac{u}{f}$.

We denote $c_1 := \inf_\Omega f$ and $c_2 := \sup_\Omega f$. Let

$$E(u) = \lambda \int |\nabla u| + \int \left(\log u + \frac{f}{u}\right) + \alpha \int \left(\frac{u}{f} - 1\right)^2$$

and

$$G(u) := \int_\Omega \left(\log u + \frac{f}{u}\right) + \alpha \int_\Omega \left(\frac{u}{f} - 1\right)^2.$$

From the proof of Theorem 3.6 in [6], for fixed $x \in \Omega$, we have

$$E(u) \ge \int_\Omega (1 + \log f).$$

Thus, $E(u)$ is bounded from below, and we can consider a minimizing sequence $\{u_n\} \in \bar{S}(\Omega)$. Fixing $x \in \Omega$, it can be easily shown that

$$g(t) := \log t + \frac{f(x)}{t} + \alpha \left(\frac{t}{f(x)} - 1\right)^2$$

is decreasing if $t \in [0, f(x))$ and increasing if $t \in (f(x), +\infty)$. It follows that one always has $g(\min(t, M)) \le g(t)$, if $M \ge f(x)$. Therefore, we have

$$G(\inf(u, c_2)) \le G(u).$$

Moreover, we have $\int_\Omega |\nabla \inf(u, c_2)| \le \int_\Omega |\nabla u|$ (see Lemma 1 in section 4.3 of [8]) and we get $E(\inf(u, c_2)) \le E(u)$. In the same way, we get $E(\sup(u, c_1)) \le E(u)$. Hence, we can assume that $0 \le c_1 \le u_n \le c_2$. This implies that u_n is bounded in $L^1(\Omega)$.

Since $\{u_n\}$ is a minimizing sequence, it follows that $E(u_n)$ is bounded. Furthermore, $\int_\Omega |\nabla u_n|$ is bounded, and $\{u_n\}$ is bounded in $BV(\Omega)$. Thus, there exists a subsequence $\{u_{n_k}\}$ which converges strongly in $L^1(\Omega)$ to some $u^* \in BV(\Omega)$, and $\{\nabla u_{n_k}\}$ converges weakly as a measure to ∇u^*. Since $\bar{S}(\Omega)$ is closed and convex, and by the lower semicontinuity of the TV term and Fatou's Lemma, we get that u^* is a solution to model (9) and necessarily, $0 < c_1 \le u^* \le c_2$. □

COROLLARY 5. *If $\alpha > \frac{1}{54}$, the solution of (9) is unique.*

PROOF. Uniqueness follows directly as a consequence of the strict convexity of the model (9). □

6. PRIMAL-DUAL METHOD

The iterative solution method we implement is based on the primal-dual method proposed by Chambolle and Pock in [4]. This algorithm, which was also used by Dong and Zeng [6], is suitable for nonsmooth convex optimization problems wherein the primal and dual formulations of a problem are combined. We first formulate the primal-dual problem associated with (9). We rewrite the TV functional in (9) as:

$$\|\nabla u\| = \max_{\|p\|_\infty \le 1} \langle p, \nabla u\rangle = -\max_{\|p\|_\infty \le 1} \langle \text{div} p, u\rangle. \tag{10}$$

which is a consequence of the Cauchy- Schwarz inequality.

Substituting (10) to (9), we can obtain the following primal-dual discretized formulation:

$$\min_{u\in X} \max_{p\in Y} J(u, p) := G(u) - \lambda \langle u, \text{div} p\rangle \tag{11}$$

where p is the corresponding dual variable, $\text{div} = -\nabla^T$ and

$$\begin{aligned}
G(u) &= \sum_{u\in S} \left(\log u + \frac{f}{u}\right) + \alpha \left(\frac{u}{f} - 1\right)^2, \\
X &= \{u \in \mathbb{R}^n : u_s \ge 0 \quad \text{for s} \in S\}, \\
S &= \{1, ..., n\} \qquad \text{is the set of pixels in the image,} \\
Y &= \{q \in \mathbb{R}^{2s} : ||q||_\infty \le 1\}, \qquad \text{and} \\
||q||_\infty &= \max_{i\in S} |\sqrt{q_i^2 + q_{i+n}^2}|
\end{aligned}$$

Note that the saddle-point problem in (11) is equivalent to interchanging the max and min functionals, i.e.,

$$\max_{p\in Y} \min_{u\in X} J(u, p) := G(u) - \lambda \langle u, \text{div} p\rangle. \tag{12}$$

We now apply the primal-dual method proposed in [4] to solve (12). This method was shown to be convergent.

The maximization problem in (13) and the minimization problem in (14) can be interpreted as an alternating primal-dual proximal point method where in a quadratic penalty term is added to force the new updates to be close to the previous ones. The equation in (15) corresponds to a simple linear extrapolation based on the new and previous updates which can be seen as an approximate extragradient step.

The maximization problem (13) can be solved directly with a gradient-ascent direction where the update solution is given by

$$p_i^{k+1} = p_i^k + \lambda \sigma (\nabla \bar{u}^k)_i \qquad \text{for} \qquad i = 1, ..., 2n. \tag{16}$$

Algorithm 1

1. Initialize $k = 0, u^0, \bar{u}^0 = u^0$, and $p^0 = (0, ..., 0)$. Fix σ, τ.

2. Calculate p^{k+1} and u^{k+1} from

$$p^{k+1} = \arg\max_{p \in Y} \lambda \langle \bar{u}^k, \mathrm{div} p \rangle - \frac{1}{2\sigma} \|p - p^k\|_2^2, \quad (13)$$

$$u^{k+1} = \arg\min_{u \in X} G(u) - \lambda \langle u, \mathrm{div} p^{k+1} \rangle + \frac{1}{2\tau} \|u - u^k\|_2^2, \quad (14)$$

$$\bar{u}^{k+1} = 2u^{k+1} - u^k \quad (15)$$

3. Stop; or set $k = k + 1$ and go to Step 2.

To ensure that the dual variable p stays feasible, a projection step is applied to (16), i.e.,

$$p_i^{k+1} = \pi_1 \left(p_i^k + \lambda \sigma (\nabla \bar{u}^k)_i \right) \qquad \text{for} \qquad i = 1, ..., 2n, \quad (17)$$

where π_1 is the projection onto the l^2-normed unit ball, i.e.,

$$\begin{aligned} \pi_1(q_i) &= \frac{q_{(i)}}{\max(1, |q_{(i)}|)}, \quad \text{and} \\ \pi_1(q_{n+1}) &= \frac{q_{(n+i)}}{\max(1, |q_{(i)}|)} \quad \text{for } i = 1, ..., n, \end{aligned}$$

with $|q_{(i)}| = \sqrt{q_i^2 + q_{i+n}^2}$.

On the other hand, the minimization problem in (14),

$$\arg\min_{u \in X} G(u) - \lambda \langle u, \mathrm{div} p^{k+1} \rangle + \frac{1}{2\tau} ||u - u^k||_2^2 := T(u) \quad (18)$$

can be easily solved by a gradient-descent method. The direction of descent is given by

$$\begin{aligned} \delta u = &\; -\nabla T(u) \\ = &\; -\frac{u - f}{u^2} - 2\alpha \left(\frac{u - f}{f^2} \right) + \lambda \;\; \mathrm{div} p^{k+1} - \frac{1}{\tau} \left(u - u^k \right) \end{aligned}$$

and the update solution is of the form

$$u^{k+1} = u^k + t \left(\frac{f - u}{u^2} - 2\alpha \left(\frac{u - f}{f^2} \right) + \lambda \mathrm{div} p^{k+1} - \frac{1}{\tau} \left(u - u^k \right) \right)$$

with a suitable steplength t. In our numerical experiments, we employ the Armijo rule to come up with a good, although inexact, steplength t.

The gradient-descent method is first order in nature since it only requires the evaluation of the objective function $T(u^k)$ and gradient $\nabla T(u^k)$ values. Although it is very simple to implement and only requires low memory, its convergence is slow. For better convergence, second order methods, in particular, Newton's method can be employed. This involves the Hessian $H = \nabla^2 T$ in the selection of the descent direction.

For the Newton's method, the direction of descent is

$$\delta u^k = -H^{-1} \nabla T(u^k), \quad (19)$$

given that H is positive-definite.

Since explicitly solving for the inverse of H in (19) is computationally expensive, it is more convenient to obtain the Newton step δu^k from solving the system

$$H \delta u^k = -\nabla T(u^k). \quad (20)$$

using preconditioned conjugate gradient (PCG) method, for instance (cf. [3]).

We now obtain a Newton step δu for the minimization problem in (18) from solving the system (20). Note that

$$\nabla T = \frac{u - f}{u^2} + 2\alpha \left(\frac{u - f}{f^2} \right) - \lambda \;\; \mathrm{div} p^{k+1} + \frac{1}{\tau} \left(u - u^k \right) \quad (21)$$

and

$$H = \nabla^2 T = \frac{2f - u}{u^3} + \frac{2\alpha}{f^2} + \frac{1}{\tau}. \quad (22)$$

with H, positive-definite for $u \in X$.

Applying the update solution (17) for (13) and the update solution for (14) obtained from the Newton step discussed above, we can now derive the specific primal-dual algorithm based on Algorithm 1 to solve (12). The algorithm is summarized as follows in Algorithm 2.

Algorithm 2

1. Initialize $k = 0, u^0, \bar{u}^0 = u^0$, and $p^0 = (0, ..., 0)$. Fix σ, τ.

2. Calculate the update solution of p^{k+1} from (17)

3. Compute the descent direction δu^k by solving

$$H \delta u^k = -\nabla T(u^k)$$

from (21) and (22).

4. Choose a step length t.

5. Calculate the update solution

$$u_{k+1} = u_k + t \delta u_k$$

6. Calculate the update solution

$$\bar{u}^{k+1} = 2u^{k+1} - u^k$$

7. Stop; or set $k = k + 1$ and go to Step 2.

Convergence of this primal-dual method proposed by

Chambolle and Pock, which we have employed to solve our proposed model, was shown in [4].

7. NUMERICS

We demonstrate by the following numerical study the capability of the proposed model (9) in reconstructing images with multiplicative noise.

All the implementations are perfomed using MATLAB R2011a. We set the same parameter values $\lambda = 1$ and $\alpha = 1$ for the DZ model and our proposed model. Quality of results of the two models are compared by means of the residual norm and peak signal-to-noise ratio (PSNR) value. The residual norm and PSNR value are computed by the equations

$$RN = \|g - h\|_2$$

and

$$PSNR = 20\log_{10}\left(\frac{1}{\sqrt{MSE}}\right)$$

where $MSE = \frac{1}{n^2}\sum_0^{n-1}\sum_0^{n-1}(g(i,j) - h(i,j))^2$ indicates the mean-squared error with $n =$ number of pixels, g as the original image and h as the restored image. Note that a lower residual norm and higher PSNR value indicate that the quality of the restored image is better.

The primal-dual algorithm is stopped when the value of the objection function has no relative decrease, i.e., when

$$\frac{E(u^k) - E(u^{k+1})}{E(u^k)} < \varepsilon. \tag{23}$$

We set $\varepsilon = 1e - 5$.

The test images are 256×256 gray level images, shown in Figure 1). The multiplicative gamma noise is generated by the MATLAB built-in function *gamrnd*.

The results for the implementations of the primal-dual method for the two models are shown in Figures 3 and 4.

Both primal-dual method implementations of the DZ model and our proposed model performed well and displayed good restoration results, in terms of noise removal and detail preservation. The restoration results of DZ displayed oversmoothening of some image details. Meanwhile, we can observe that although noise relics are present in our restoration results, oversmoothing was least manifested.

We can also observe that our model has a better qualitative performance compared to DZ. Table 1 shows the residual norms (*ResNorm*), PSNR values and number of iterations (*Iter*) for the two models. The primal-dual method implementation of our model outperforms that of DZ, since lower residual norms, higher PSNR values and lower number of iterations can be observed from our restoration results.

(a) Cameraman

(b) Peppers

Figure 1: Original Images

8. CONCLUSION

In this paper, theoretical features of the variation model in [5] were analysed. Mean values for the random variable $Y = 1/\eta$ are derived and the strict convexity of the model was discussed. In addition, the model was shown to admit a unique solution. Further research could dwell on dealing with noise with mean not equal to one. The stopping rule can also be enhanced by considering Karush-Kuhn-Tucker conditions.

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(a) K=20

(b) K=50

Figure 2: Noisy Images corrupted by multiplicative gamma noise

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(a) DZ model

(b) Our model

Figure 3: Results of primal-dual method when removing the multiplicative noise in the noisy Cameraman image with $K = 20$.

(a) DZ model

(b) Our model

Figure 4: Results of primal-dual method when removing the multiplicative noise in the noisy Peppers image with $K = 50$.

Images	Model	*ResNorm*	PSNR	*Iter*
Cameraman	DZ	17.92	20.96	108
	Ours	15.63	22.04	89
Peppers	DZ	10.42	24.61	87
	Ours	7.37	27.21	71

Table 1: Residual norm, PSNR values and # of iterations for the two images using the primal- dual method of the two different models

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