On Maximizing the Present Value of Future Dividends Using Stochastic Control

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ABSTRACT

We illustrate the use of the Hamilton Jacobi Bellman (HJB) equation in solving a stochastic control problem. The method involves transforming the stochastic differential equation into a simpler form, but techniques in solving ODEs still have to be used to obtain the closed form of the solution. We also present a simple model of an insurance company whose wealth is invested in risky and risk-free assets, claims are assumed to follow a Brownian motion and proportional reinsurance is used to transfer some risk to the reinsurer. The wealth is controlled by the ratio of claims retained by the company, the proportion of wealth invested in risky and risky-free assets, and in the amount of dividend paid out. The reward function is the expected present value of future dividends and we solve the stochastic control problem using the above mentioned method. We present closed form solutions for the optimal controls and optimal reward functions.

Keywords: stochastic control, Hamilton Jacobi Bellman equation, insurance

1. Introduction

In this thesis, we will present a method to solve a stochastic control problem using the Hamilton Jacobi Bellman equation. This is already a classical method and has been applied to a lot of practical problems, especially in finance and engineering [4, 1]. Obtaining closed form solutions is not possible for a lot of these types of problems, and one of our goals is to be able to present an application of the theory in a stochastic problem that allows a closed form solution.

We will first discuss some basic concepts stochastic control, and illustrate the use of the Hamilton Jacobi Bellman Equation in solving stochastic control problem. We will then present two examples where stochastic control theory could be applied. A classical example would be followed by an example where we will use results in stochastic control to control the wealth of an insurance company. We will present a simplified stochastic differential equation describing

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the wealth of the company but the wealth that we will consider here, as discussed in Section 4, is simplified to make our analysis of the control problem feasible. Our model of the insurance company involves investments in risky and risk-free assets. Inflow of funds are accredited to premium collection, and outflow of funds are accredited only to insurance benefits/claims and dividends disbursed by the company to its stockholders.

In Section 2, we will briefly present some preliminary concepts of stochastic control. We will then present a classical example of stochastic control in Section 3 and then, in Section 4 we give another example of stochastic control applied to an insurance company. Finally, we summarize the concepts we used in Section 5.

2. Stochastic Control

In this section, we will discuss Stochastic Control as developed by Hojgaard [1]. In stochastic control theory, we are concerned with a stochastic process $\{X_t\}$ which we could influence or control via a control functional $u(t)$.

We also say that the control $u(t)$ is admissible if it is adapted to the filtration $\{\mathcal{F}_t\}$. This means that $u(t)$ is $\mathcal{F}_t$-measurable for all t. We define U to be the set of all admissible controls. We will consider a controlled process governed by the stochastic differential equation

$$dX_t = \mu\,(\,t, X_t, u(t)\,)\,dt + \sigma\,(\,t, X_t, u(t)\,)\,dB_t, \quad t > t_0 \tag{2.1}$$

$$X_{t_0} = x_0 \tag{2.2}$$

where X_t, t_0 and x_0 denote the random state process, the initial time and the initial state, respectively. For each admissible control u we introduce a reward functional,

$$V^u(t,x) = E_{t,x}\left[\int_t^T h\,(s, X_s, u(s, X_s)\,)\,ds + g\,(X_T)\right] \tag{2.3}$$

where $t_0 \le t \le T$, T is a finite constant and $x = X_t$. The objective is then to find the optimal reward function

$$V(t,x) = \sup_{u \in U} V^u(t,x). \tag{2.4}$$

We will now state some important theorems which will enable us to explicitly determine V and u, the optimal reward function and optimal control, respectively.

Theorem 1 (Hamilton-Jacobi Bellman equation (HJB)) *With respect to the above control problem, Equations (2.1)-(2.4), assume $V \in C^{1,2}\,(R \times R)$. Then V satisfies the HJB equation*

$$\sup_{u \in U}\,[\,\mathcal{L}^u V(t,x) + h(t,x,u)\,] = 0 \tag{2.5}$$

where

$$\mathcal{L}^u V(t,x) = \frac{\partial}{\partial t}V(t,x) + \mu(t,x,u)\,\frac{\partial}{\partial x}V(t,x) + \frac{1}{2}\sigma^2(t,x,u)\,\frac{\partial^2}{\partial x^2}V(t,x) \tag{2.6}$$

with terminal condition

$$V(T,x) = g(x). \tag{2.7}$$

The smoothness requirement, $V \in C^{1,2}(R \times R)$ is hard to verify beforehand. But Theorem 1 is still useful to us due to the following theorem which states that any solution of the HJB equation yields the optimal reward function [1].

Theorem 2 (Verification theorem) *With respect to the above control problem, Equations (2.1)-(2.4), assume $Z \in C^{1,2}(R \times R)$ is a solution of the HJB equation (2.5) with terminal condition (2.7). Let*

$$u^*(t,x) = argmax\left[\mathcal{L}^u Z(t,x) + h(t,x,u)\right] \tag{2.8}$$

satisfying

$$\mathcal{L}^{u^*} Z(t,x) + h(t,x,u^*) = 0. \tag{2.9}$$

Then the optimal process X_t^* satisfies Equation (2.1) with $u(t) = u^*(t, X^*(t))$ and $Z(t,x) = V(t,x) = V^{u^*}(t,x)$.

Remark 1 Theorem 2 also gives us the form of the optimal control u^* in terms of the solution of the HJB equation. Moreover the optimal control u^* is a feedback control [1].

We now extend our discussion of stochastic differential equations with more than one standard Brownian motion. We thus extend Equation (2.1) into the form

$$dX_t = \mu\,(t, X_t, u(t))\,dt + \sum_{i=1}^{n} \sigma_i\,(t, X_t, u(t))\,dB_t^{(i)}, \qquad t > t_0 \tag{2.10}$$

$$X_{t_0} = x_0, \tag{2.11}$$

where the $\left\{B_t^{(i)}\right\}$ are pairwise independent standard Brownian motion. We will also extend our discussion of stochastic control to systems governed by the stochastic differential equations of the form given in Equation (2.10). We will use the same reward function in Equation (2.3) and the same optimal reward function in Equation (2.4). In order to find V and u we still make use of Theorem 1 with

$$\mathcal{L}^u V(t,x) = \frac{\partial}{\partial t} V(t,x) + \mu(t,x,u)\,\frac{\partial}{\partial x} V(t,x) + \frac{1}{2}\left(\sum_{i=1}^{n} \sigma_i^2(t,x,u)\right)\frac{\partial^2}{\partial x^2} V(t,x). \tag{2.12}$$

The Verification Theorem i.e., Theorem 2, would still hold with $\mathcal{L}^u V(t,x)$ defined in Equation (2.12). This extension of the stochastic control would be employed in solving our control problem in Section 4.

3. Classical Example

To illustrate how to solve a stochastic control problem using the concepts we presented in Section 2, we give a simple example by Hojgaard [1].

Example 1 Let X_t denote the position of a controlled device subject to random disturbances and assume it is governed by the stochastic differential equation.

$$\begin{cases} dX_t = u(t)dt + \sigma dB_t \\ X_0 = x_0. \end{cases}$$

where $u(t) \in R$ is the control functional. The objective is to minimize

$$E_{0,x_0} \int_0^T au^2(s) + b(x - x_1)^2(s)ds, \quad a, b > 0.$$

To this end we define,

$$V^u(t, x) = E_{t,x} \int_t^T au^2(s) + b(x - x_1)^2(s)ds, \quad a, b > 0. \tag{3.13}$$

Hence we want to find out

$$V(t, x) = \inf_{u \in U} E_{t,x} \int_t^T au^2(s) + b(x - x_1)^2(s)ds$$

where $0 \leq t \leq T$, $T < \infty$ is a constant and $x = X_t$. In particular we want to find out $V(0, x_0)$.

As seen in the reward functional in Equation (3.13), we want to regulate the position to a desired state x_1 while not using too much energy to control the position. The constants a and b are weights we can use to give more importance to using minimal control u, or to regulate the state X_t.

The resulting HJB equation is

$$0 = \inf_{u \in U} \left\{ V_t + \frac{1}{2}\sigma^2 V_{xx} + uV_x + \left[a\, u^2 + b(x - x_1)^2 \right] \right\}.$$

We differentiate the preceeding equation with respect to u in order to get u^* which would minimize the HJB equation. The minimizer is

$$u^*(t, x) = -\frac{1}{2a} V_x.$$

We substitute this expression for u^* back to the HJB equation in order to get an expression solely in terms of V. We get

$$V_t + \frac{1}{2}\sigma^2 V_{xx} - \frac{1}{4} V_x^2 + b(x - x_1)^2 = 0. \tag{3.14}$$

We now use a technique adopted from [1] wherein the form of the solution is motivated by the deterministic control. The form of the solution that we will use is

$$V(t, x) = P(t)\,(x - x_1)^2 + Q(t).$$

Substituting this to Equation(3.14), we obtain

$$P'(t) \cdot (x - x_1)^2 + Q'(t) + \sigma^2 P(t) - \frac{1}{4}\left[2(x - x_1) \cdot P(t)\right]^2 + b(x - x_1)^2 = 0.$$

In order for the previous equation to hold, we separate those terms with $(x - x_1)^2$ and those without and equate coefficients to zero. This yields

$$P'(t) - \frac{1}{a} P^2(t) + b = 0 \tag{3.15}$$

and

$$Q'(t) = -\sigma^2 P(t). \tag{3.16}$$

From the definition of V we could infer that the terminal condition is $V(T, x) = 0$. This implies $P(T) = Q(T) = 0$. We can solve the differential equations (3.15) and (3.16) by employing their terminal conditions and we obtain

$$P(t) = \sqrt{ab}\, \tanh \left[(T - t) \sqrt{\frac{b}{a}} \right]$$

and

$$Q(t) = \sigma^2 b \log \left(\cosh \left[(T-t) \sqrt{\frac{b}{a}} \right] \right).$$

The optimal control function is therefore

$$u^*(t,x) = -\frac{1}{2a} V_x(t,x) = -\frac{P(t)}{a}(x - x_1).$$

The conclusion is to push the particle towards x_1 with a force proportional $\frac{P(t)}{a}$ which approaches 0.

4. Motivating Example: Insurance Company

Motivated by an example in Fleming [4] we will now develop a model of an insurance company. Let X_t denote the wealth of the insurance company at any $t \geq 0$. Initially X_0 is what could be invested in a risky or risk-free asset. Then at $t > 0$, X_t is incremented by the returns on investments and premiums and is decremented by the dividends given out and claims paid.

We will assume that the insurance company has only two investment opportunities: a risk-free investment and a risky investment. We will further assume that the company invests all its wealth in the risky and/or risk-free asset and we will also assume that the withrawal of funds from both assets are instantaneously possible to cover the insurance claims. We would like to point out that the "wealth" we consider in this example might not include other amounts that are usually included in practice.

Let $b(t)$ be the proportion of the wealth of the company invested in the risky asset and $1-b(t)$ be the proportion of the amount invested at the risk-free asset. Then if we let F_t be the amount of investment on the riskless asset at any time $t \geq 0$ and assume that this investment is risk-free with risk-free rate $r_0 > 0$, then F_t can be modeled as

$$dF_t = [1 - b(t)]\, X_t\, (r_0\, dt). \tag{4.17}$$

Let Z_t be the amount of investment on the risky asset at any time $t \geq 0$. We model this investment using a Brownian motion, hence it can be written as

$$dZ_t = b(t)\, X_t \left(r_1\, dt + \sigma_p dB_t^{(1)} \right), \tag{4.18}$$

where $r_1, \sigma_p^2 t > 0$ and $B_t^{(1)}$, are the expected return from the risky asset, the variance of the return from the risky asset at any time $t \geq 0$, and standard Brownian motion with respect to the filtration $\{\mathcal{F}_t\}$. Note that $r_1 > r_0$ because investors would require this due to the inherent volatility of the risky asset.

We now incorporate in our model the very nature of an insurance company. We need to account for the claims going out and the premiums coming in. We assume that the premiums come regularly while the insurance claims are random in amount and occurs randomly. Let R_t be the cummulative summation of the premiums received less the claims being paid out at time $t \geq 0$. As discussed by Klugman [3] one acceptable way to describe R_t is to assume that it follows a Brownian motion. Hence we have the stochastic differential equation,

$$dR_t = \mu\, dt + \sigma dB_t^{(2)} \tag{4.19}$$

where μ, $\sigma^2 t > 0$ and $B_t^{(2)}$ are the expected level of R_t, the variance of R_t at any time $t \geq 0$, standard Brownian motion with respect to the filtration $\{\mathcal{F}_t\}$. $B_t^{(2)}$ is independent of $B_t^{(1)}$.

We will consider a proportional reinsurance. Let $a(t)$ be the fraction that corresponds to the proportion of claims and premiums retained by the company. This implies that the company reinsures $1 - a(t)$ of the claims. As an example, if there is a premium payment of 100 and $a = 0.70$ then the insurance company will keep 70 and give the reinsurer 30. And from a claim amount of 1000, 700 would be shouldered by the insurance company and 300 will be shouldered by the reinsurer.

Let $l(t)$ be the amount of dividends disbursed by the insurance company at time t. The expected present value of future dividends is what we want to maximize, hence our reward function will only involve $l(t)$.

Combining the different parts of our model we arrive at the following stochastic differential equation

$$\begin{cases} dX_t = dZ_t + dF_t + a(t)\, dR_t - l(t)dt \\ \\ X_0 \;= x_0. \end{cases} \tag{4.20}$$

where x_0 is the initial wealth of the company.

We can control the stochastic random variable X_t in Equation (4.20) by varying the values of a, b and l. A control policy u is described by an ordered triple

$$u(t) = \Big(a_u(t),\, b_u(t),\, l_u(t) \Big)$$

of stochastic processes. Note that the subscript u indicates a specific combination of the control variables. When applying the policy u we denote by X_t^u the resulting optimal wealth of the company, which is modeled by

$$\begin{cases} dX_t^u = dZ_t^u + dF_t^u + a_u(t)\, dR_t - l_u(t)dt \\ \\ X_0^u \;= x_0. \end{cases} \tag{4.21}$$

where x_0 is the initial wealth of the company. Incorporating Equations (4.17), (4.18) and (4.19) in (4.21), we arrive at

$$\begin{cases} dX_t^u = [\, \mu \cdot a_u(t) - l_u(t) + X_t^u \,(r_0 + b_u(t)\, \{r_1 - r_0\}\,)\,]\, dt \\ \\ \qquad\qquad + b_u(t)\, X_t^u\, \sigma_p\, dB_t^{(1)} + a_u(t)\, \sigma\, dB_t^{(2)} \\ \\ X_0^u \;= x_0. \end{cases} \tag{4.22}$$

Equation (4.22) is the model of an insurance company that we will consider.

4.1. The Control Problem and the Hamilton-Jacobi-Bellman Equation

We will now try to obtain the control policy u that will cause the company's wealth to be optimized according to a criterion that we will specify. We are interested in maximizing the value or worth of the company by maximizing the common stock of the insurance company. One way to maximize the common stock of the company is to maximize the present value of future dividends. To this end, we initially set our reward function as

$$\widehat{V}^u(t,x) = E_{t,x} \int_t^T e^{-\delta s}\, [l_u(s)]\, ds \tag{4.23}$$

where $0 \le t \le T$, with $T < \infty$ a constant, and $\delta > 0$ is the force of interest. $V^u(t,x)$ is the expected continuous present value of future dividends until a specified future time T. However when we tried to solve the problem we were not able to come up with a closed form solution. Hence, we set our reward function as

$$V^u(t,x) = E_{t,x} \int_t^T e^{-\delta s} \left[l_u(s)\right]^n ds \tag{4.24}$$

where $0 < n < 1$ .Maximizing Equation (4.23) is equivalent to maximizing Equation (4.24) since $[l_u(t)]^n$ is just a concave function. The objective then is to find the optimal return function defined by

$$V(t,x) = \sup_{u \in U} V^u(t,x), \tag{4.25}$$

and to find an optimal policy u^* that satisfies $V^{u^*}(t,x) = V(t,x)$ for all initial time t and all initial wealth x.

Note that $V(t,0) = 0$ because if the company starts with a wealth of $x = 0$ then no dividends could be given out. Furthermore, $V(T,x) = 0$ because time T is the end of the process we are considering and hence the company will not be able to give dividends at the end of the time frame.

Applying the HJB equation with respect to (4.25) subject to the dynamics of (4.22) we obtain

$$\sup_{u \in U} \left\{ V_t + \frac{1}{2}\left(a^2\sigma^2 + b^2x^2\sigma_p^2\right) V_{xx} + \left(a\mu - l + x\left[r_0 + b(r_1 - r_0)\right]\right) V_x + e^{-\delta t}[l(t)]^n \right\} = 0 \tag{4.26}$$

We set the partial derivatives of Equation(4.26) with respect to the three control variables, a, b and l to zero. We obtain

$$\begin{aligned} b(t,x) &= \frac{-(r_1 - r_0)\, V_x}{x\,\sigma_p^2\, V_{xx}} \\ a(t,x) &= \frac{-\mu\, V_x}{\sigma^2\, V_{xx}} \\ l(t,x) &= \left(\frac{V_x\, e^{\delta t}}{n}\right)^{\frac{1}{n-1}}. \end{aligned}$$

We will now use the strategy used in [1] wherein the form of the solution V is assumed. We tried different forms of the solution, and we found out that the form $V(t,x) = g(t)\, x^n$ works. This form is motivated by deterministic control and incorporating this conjectured solution to Equations (4.1), (4.1) and (4.1) we obtain

$$\begin{aligned} b(t,x) &= \frac{r_1 - r_0}{\sigma_p^2\,(1-n)} \\ a(t,x) &= \frac{\mu\, x}{\sigma^2\,(1-n)} \\ l(t,x) &= x\left[e^{\delta t}\, g(t)\right]^{\frac{1}{n-1}}. \end{aligned}$$

We now substitute these maximizers to the HJB equation (4.26) to arrive at

$$
\begin{aligned}
0 &= g'(t)\, x^n \\
&+ \frac{1}{2}\left\{ \left(\frac{x^2\, \mu^2}{\sigma^4 \cdot (1-n)^2} \right) \sigma^2 + \left(\frac{x^2\, (r_1 - r_0)^2}{\sigma_p^4 \cdot (1-n)^2} \right) \sigma_p^2 \right\} \left[g(t)\, n\, (n-1) x^{n-2} \right] \\
&+ \left\{ \left(\frac{\mu^2\, x}{\sigma^2\, (1-n)} \right) - \left[e^{\delta t}\, g(t) \right]^{\frac{1}{n-1}} x + x \left[r_0 + \left(\frac{(r_1 - r_0)^2}{\sigma_p^2\, (1-n)} \right) \right] \right\} \left[g(t)\, n\, x^{n-1} \right] \\
&+ e^{-\delta t} \left\{ \left[e^{\delta t}\, g(t) \right]^{\frac{1}{n-1}} x \right\}^n .
\end{aligned}
$$

Simplifying the equation above yields

$$
0 = x^n \left\{ g'(t) + n \left(r_0 + \frac{(r_1 - r_0)^2}{2\sigma_p^2\, (1-n)} + \frac{\mu^2}{2\sigma^2\, (1-n)} \right) g(t) + (1-n) \left[e^{\delta t}\, g(t) \right]^{\frac{1}{n-1}} g(t) \right\} .
$$

To simplify the notations, we let $\alpha = r_0 + \dfrac{(r_1 - r_0)^2}{2\sigma_p^2\, (1-n)} + \dfrac{\mu^2}{2\sigma^2\, (1-n)}$. Note also that $g(T) = 0$ since $V(T, x) = 0$. Hence $g(t)$ satisfies the ordinary differential equation

$$
\begin{cases}
0 = g'(t) + n\alpha\, g(t) + (1-n) \left[e^{\delta t}\, g(t) \right]^{\frac{1}{n-1}} g(t) \\
\\
0 = g(T).
\end{cases}
$$

We make the transformation $h(t) = \left[e^{\delta t} \cdot g(t) \right]^{\frac{1}{1-n}}$ and solve for $h(t)$ in the transformed ordinary differential equation

$$
\begin{cases}
0 = h'(t) + \left[\dfrac{n\alpha - \delta}{1-n} \right] h(t) + 1 \\
\\
0 = h(T).
\end{cases}
\tag{4.27}
$$

The solution is

$$
h(t) = \left(\frac{1-n}{\delta - n\alpha} \right) \left[1 - e^{\frac{-(\delta - n\alpha)\,(T-t)}{1-n}} \right] . \tag{4.28}
$$

Consequently

$$
g(t) = e^{-\delta t} \left\{ \left(\frac{1-n}{\delta - n\alpha} \right) \left[1 - e^{\frac{-(\delta - n\alpha)\,(T-t)}{1-n}} \right] \right\}^{1-n} . \tag{4.29}
$$

This implies that our assumption on the form of V, i.e., $V(t, x) = g(t) x^n$ is justified. It is evident from Equation (4.29) that $V(t, x) \in C^{1,2}\,(R \times R)$, hence by Theorem 2, this is the solution that we are looking for.

4.2. Analysis of the Control Variables

We now examine computed optimal control functions a, b and l from the previous subsection.

4.2.1. The Investment Proportion b

We know that

$$b(t,x) = \frac{r_1 - r_0}{\sigma_p^2 (1-n)}. \tag{4.30}$$

We first point out that if the difference between the risk-free rate, r_0, and the expected return of the risky asset, r_1, is small then a large proportion would be invested in the risk-free asset rather than in the risky asset. However if the difference is large then the opposite would be true. If σ_p^2 is small this means that the risky asset would not fluctuate much and hence a larger proportion would be invested in the risky asset. Looking at n, if n is near 1 then we invest a larger proportion to the risky asset. On the other hand if n is near zero then a larger proportion would be invested in the risk-free asset. Note that $b(x,t)$ is independent of x and t and is just a constant. This means that the proportion to be invested in the risky asset is just the same no matter how much wealth the company has. This could be viewed as the company using a safe investment strategy. Lastly, if in the computation of $b(x,t)$ the answer is greater than 1, we set it equal to 1. This is because the company can only invest until a 100% of its wealth in the risky asset and because we do not allow short sales. Hence we have the maximizer

$$b^*(t,x) = \begin{cases} \dfrac{r_1 - r_0}{\sigma_p^2 (1-n)}, & r_1 - r_0 < \sigma_p^2 (1-n) \\ 1 & \text{otherwise.} \end{cases} \tag{4.31}$$

We now look at some simulations of b^*. Suppose our specific values are as follows, $r_1 = 8\%$, $r_0 = 5\%$, $\delta = 10\%$, $n = 0.99$ and $T = 5$. In Figure 1 the dotted lines correspond to b with $\sigma_p^2 = 0.10$. The computed value of $b(t,x)$ is 3.75 hence $b^*(t,x) = 1$. Also in Figure 1, the solid line correspond to $b(x,t)$ with $\sigma_p^2 = 0.03$. The computed value of $b(t,x)$ is 0.75 and hence $b^*(t,x) = 0.75$.

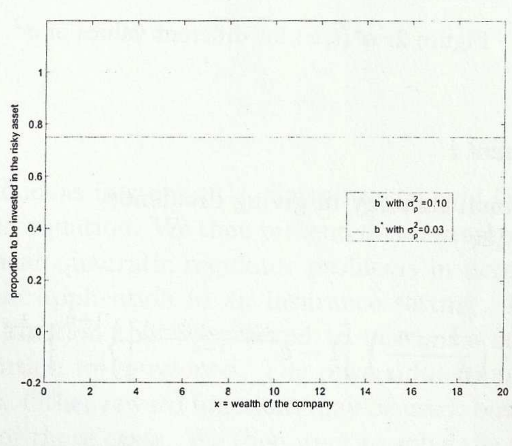

Figure 1: $b^*(t,x)$ for different values of σ_p^2.

4.2.2. The Reinsurance Proportion a

We next consider the proportion that will be reinsured, assuming a proportional reinsurance,

$$a(t,x) = \frac{\mu\, x}{\sigma^2\,(1-n)}. \tag{4.32}$$

We first point out that, $a(t,x)$ is an increasing function of x and is independent of t. If the initial wealth is below the treshold wealth of $\dfrac{\sigma^2\,(1-n)}{\mu}$ then the company will reinsure part of the claims and give part of the premiums to the reinsurer. However, once the company's wealth is above this threshold then the company can fully shoulder the claims of its policyholders. This means that if $a(t,x) \geq 1$, then we set $a^*(t,x) = 1$.

$$a^*(x,t) = \begin{cases} \dfrac{\mu\, x}{\sigma^2 \cdot (1-n)}, & x < \dfrac{\sigma^2\,(1-n)}{\mu} \\ 1 & \text{otherwise.} \end{cases} \tag{4.33}$$

We now look at some simulations of a^*. Suppose our specific values are as follows, $\mu = 4000$ and $n = 0.99$. In Figure 2 the dotted lines correspond to $a^*(t,x)$ with $\sigma = 1500$ while the solid line correspond to $a^*(t,x)$ with $\sigma = 2500$.

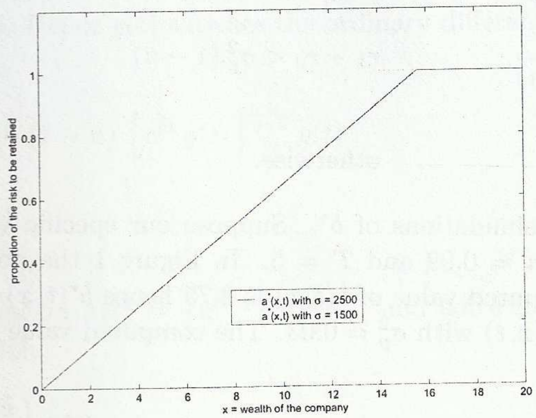

Figure 2: $a^*(t,x)$ for different values of σ^2.

4.2.3. Dividend disbursement l

Lastly we analyze the optimal strategy in giving dividends,

$$\begin{aligned} l(x,t) &= x\left[e^{\delta t} \cdot g(t)\right]^{\frac{1}{n-1}} \\ &= x\left[e^{-\delta t} \cdot e^{\delta t}\left\{\left(\frac{1-n}{\delta - n\alpha}\right)\left[1 - e^{\frac{-(\delta - n\alpha)\cdot(T-t)}{1-n}}\right]\right\}^{1-n}\right]^{\frac{1}{n-1}} \\ &= \frac{x}{\left(\dfrac{1-n}{\delta - n\alpha}\right)\cdot\left(1 - e^{\frac{-(\delta-n\alpha)\cdot(T-t)}{1-n}}\right)}. \end{aligned}$$

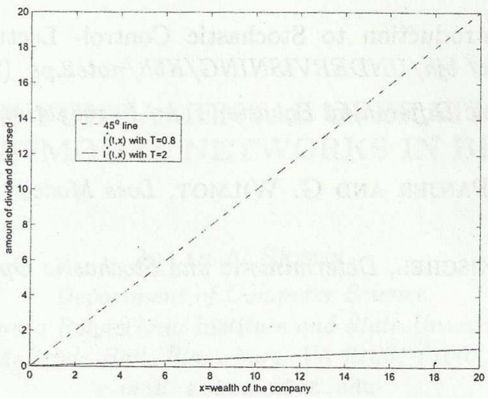

Figure 3: $l^*(t,x)$

Since we want the dividends to be less than the wealth of the company we have,

$$l^*(x,t) = \begin{cases} \dfrac{x}{\left(\dfrac{1-n}{\delta - n\alpha}\right)\cdot\left(1 - e^{\frac{-(\delta-n\alpha)\cdot(T-t)}{1-n}}\right)}, & l(x,t) < x \\ \\ x & \text{otherwise.} \end{cases} \tag{4.34}$$

For a fixed t, the company disburses a constant proportion of its wealth to its stockholders. Moreover, if $T-t$ is large(small) then l^* is small(large). This would imply that a large time span would result in smaller amounts of dividends being continuously paid as compared to what would be disbursed had the time span been small.

We now look at a simulation of l^*. Suppose our specific values are as follows, $\mu = 20$, $\sigma = 20$, $r_1 = 8\%$, $r_0 = 5\%$, $\sigma_p^2 = .03$, $\delta = 0.10, n = 0.80$ and $t = 0$. In Figure 3 the dashed line correspond to the 45^o line, the dotted line correspond to $l^*(t,x)$ with $T = 0.8$ while the solid line correspond to $l^*(t,x)$ with $T = 2$.

5. Summary

We have presented the basic ideas in stochastic control theory and illustrated its solution using the Hamilton Jacobi Bellman equation. We then presented two examples, one a classical example that is analogous to the linear quadratic regulator problems in deterministic control, and the other one, an example of an application in an insurance setting. In the insurance example, we formulated the reward function that we wanted to maximize subject to the dynamics of a stochastic differential equation we developed. The reward function was the expected present value of the future dividends. Other reward functions may be used, but the method we considered here may or may not work for those cases. We then went to solve the stochastic control problem using a classical method: the Hamilton-Jacobi-Bellman equation. The control variables we obtained from the HJB equation were analyzed subject to the constraints of the insurance company.

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