On the Seat Allocation Error and the Principle of Proportional Representation (3rd draft)

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Abstract

By the principle of proportional representation for the party-list system in a multi-seat election where a voter votes for only one party, a party-list organization that obtains $n\%$ of the total votes for the party-list is entitled to $n\%$ of the total seats. The seat allocation error of seat allocation method on a party's number of votes in the set of votes in the party-list election is the difference between the ideal number of seats that a party wins based on the principle of proportional representation and the actual number of seats allocated to the party by the seat allocation method. The seat allocation method affirms the proportionality principle on a party's number of votes if the absolute value of the seat allocation error is less than one. Otherwise, the seat allocation method negates the proportionality principle on the party's number of votes. In this paper, we shall show that largest remainder method affirms the proportionality principle, but the highest average method may negate in some instances. We shall also show that the seat allocation method of the Philippine Party-List Law (RA 7941) negates the principle of proportionality representation.

Keywords: Seat Allocation Method, Principle of Proportional Representation, Seat Allocation Error, Highest Average Method, Largest Remainder Method.

1 Introduction

In a multi-seat election of a party-list system where a voter chooses only one party from a list of registered parties, the problem in allocating the specified number of seats to the registered parties in accordance with the principle of proportional representation arises.

The party-list proportional system are used by about 62 countries worldwide. In this system the seats are allocated according to the share of the total votes cast for the party-list that each party has received. In determining the number of seats that will be allocated to the parties, the highest average ('d Hondt) method or the largest remainder (Hare) method are used. See the formulas in [3].

1. In the highest average method, the votes for each party are to be divided by a series of divisors. The seats are allocated based on the largest quotient up to the number of seats available.

2. In the largest remainder method, the allocation of seats consists of two rounds. The first round computes the automatic number of seats that a party. It is based on the integral part of the quotient when the number of votes received by the party is divided by a minimum quota. The second round distributes the remaining number of seats not allocated by the first round. The remainders are ranked from the highest to the lowest. The parties with the highest ranked remainders win one seat up to the number of seats available.

Example 1 (Highest Average Method)
Suppose that there are 4 parties p_1, p_2, p_3, p_4 registered in a party-list election where the respective votes are $v_1 = 5,700$, $v_2 = 2,700$, $v_3 = 1,200$ and $v_4 = 400$ and the specified number of seats is 5.
Using the Highest Average method, we divide the votes by $1, 2, 3, 4$, to obtain
The 5 highest quotients are:

1. $5,700$ of p_1,
2. $2,850$ of p_1,
3. $2,700$ of p_2,
4. $1,900$ of p_1,
5. $1,425$ of p_1.

Table 1: The Highest Average Method

	Divisors			
p_i	1	2	3	4
p_1	$5,700$	$2,850$	$1,900$	$1,425$
p_2	$2,700$	$1,350$	900	675
p_3	$1,200$	600	400	300
p_4	400	200	$133.\bar{3}$	100

Therefore, by the **HA** method the 5 specified seats will be allocated in the following manner: 4 seats for party p_1, 1 seat for party p_2, and none for p_3 and p_4.

Example 2 (Largest Remainder Method)
Using the data in Example 1, we construct the following table. Note that 1 seat is equivalent to $\frac{10,000}{5} = 2,000$ votes which is the minimum quota.

Table 2: The Largest Remainder Method

p_i	v_i	int $\left(\frac{v_i s_T}{v_T}\right)$	Remainder	Rank	Addt'l Seat	Total Seats
p_1	$5,700$	2	$1,700$	1	1	3
p_2	$2,700$	1	700	3	0	1
p_3	$1,200$	0	$1,200$	2	1	1
p_4	400	0	400	4	0	0
	10,000	3			2	5

Therefore, by the **LR** method the 5 specified seats will be allocated in the following manner: 3 seats for party p_1, 1 seat for party p_2, 1 seat for p_3 and none for p_4.

We produce the following algorithms for the Highest Average Method and the Largest Remainder Method.

Algorithm 1 (Highest Average Method)
Input: $(p_1, v_1), (p_2, v_2), \ldots, (p_n, v_n)$ and s_T.
Output: $(p_1, s_1), (p_2, s_2), \ldots, (p_n, s_n)$

Step 1. Form the set $Q = \{v_i/j \mid i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, s_T\}$ and for $i = 1$ to n, $s_i = 0$.

Step 2. For each i and j count the number m of elements in Q that is greater than or equal to v_i/j.

If $m < s_T$, then v_i/j adds one seat for p_i.
Otherwise, v_i/j adds no seat for p_i.

Step 3. Sum up all the seats of p_i, let this be equal to s_i, for $i = 1, 2, \ldots, n$.

Step 4. Return (p_i, s_i) for $i = 1, 2, \ldots, n$.

Algorithm 2 (Largest Remainder Method)
Input: $(p_1, v_1), (p_2, v_2), \ldots, (p_n, v_n)$ and s_T.
Output: $(p_1, s_1), (p_2, s_2), \ldots, (p_n, s_n)$

Step 1. Find the sum $v_t = v_1 + v_2 + \ldots + v_n$.

Step 2. For $i = 1$ to n, compute $(s_i, r_i) = \text{divmod}(v_i * s_T, v_T)$.

Step 3. Find $d = s_T - s_1 - s_2 - \ldots - s_n$.

Step 4. Rank each r_i's and if $\text{rank}(r_i) \leq d$, then $s_i = s_i + 1$, for $i = 1, 2, \ldots, n$.

Step 5. Return (p_i, s_i) for $i = 1, 2, \ldots, n$.

2 The Principle of Proportional Representation

The principle of proportional representation may be stated as follows:

A party that obtains $n\%$ of the total votes shall be awarded $n\%$ of the total seats.

Let v_T and s_T be the total number of votes for the party-list and s_T be the available number of seats.

Theorem 1
If a party obtains v number of votes, then it shall be given $\dfrac{vs_T}{v_T}$ seats.

Proof.
The party receives $\dfrac{v}{v_T}$ of the total number of votes. Hence, by the principle of proportional representation, it must be awarded $\dfrac{v}{v_T} \times s_T$ of the total number of seats. □
Since most of the time vs_T is not divisible by v_T, then there is an error in the allocation of seat. Hence, we have the following definition.

Definition 1 (Seat Allocation Error)
Let F be the seat allocation method applied in a party-list election. Suppose that a party receives v votes and $F(v)$ is the number of seats allocated by the method. Then the *seat allocation error* of F on v denoted by $\text{error}_{F(v)}$ is given by

$$\text{error}_{F(v)} = \frac{vs_T}{v_T} \tag{1}$$

In our previous examples, p_1 obtains $5,700$ votes out of $10,000$ votes for the party-list and there 10 available seats. By the HA method p_1 receives 4 seats. However, by the LR method p_1 receives 3 seats.
The seat allocation error of HA on $v_1 = 5,700,\ v_2 = 2,700,\ v_3 = 1,200,\ v_4 = 400$:

1. $\text{error}_{HA(v_1)} = \dfrac{(57000)(5)}{10000} - 4 = 2.85 - 4 = -1.15.$
2. $\text{error}_{HA(v_2)} = \dfrac{(27000)(5)}{10000} - 1 = 1.35 - 1 = 0.35.$
3. $\text{error}_{HA(v_3)} = \dfrac{(12000)(5)}{10000} - 0 = 0.6 - 0 = 0.6.$
4. $\text{error}_{HA(v_4)} = \dfrac{(400)(5)}{10000} - 0 = 0.2 - 0 = 0.2.$

and the seat allocation of LR on $v_1 = 5,700,\ v_2 = 2,700,\ v_3 = 1,200,\ v_4 = 400$:

1. $\text{error}_{LR(v_1)} = \frac{(57000)(5)}{10000} - 3 = 2.85 - 3 = -0.15.$
2. $\text{error}_{LR(v_2)} = \dfrac{(27000)(5)}{10000} - 1 = 1.35 - 1 = 0.35.$
3. $\text{error}_{LR(v_3)} = \dfrac{(12000)(5)}{10000} - 1 = 0.6 - 1 = -0.4.$
4. $\text{error}_{LR(v_4)} = \dfrac{(400)(5)}{10000} - 0 = 0.2 - 0 = 0.2.$

We have the following definition.

Definition 2
Let $|\text{error}_{F(v)}| = \epsilon$. The seat allocation method F affirms the principle of proportional representation on $v \in V$ if $\epsilon < 1$. Otherwise if $\epsilon \geq 1$, F negates the principle of proportional representation on v and the degree of negation of F on $v \in V$ is equal to $\text{int}(\epsilon)$ seats where $\text{int}(\epsilon)$ is the integer part of ϵ.
If *epsilon* < 1 for all $v \in V$, then we say that F affirms the principle of proportional representation on V. Otherwise, F negates the principle of proportional representation on V where the degree of negation is

$$\sum_{v \in V} \text{int}\left(|\text{error}_{F(v)}|\right).$$

The degree of negation is called the *method-induced error.*

Definition 3
Let $V = \{v_1, v_2, \ldots, v_n\}$ be the set of votes in a party-list election. If $\epsilon_i = |\text{error}_{F(v_i)}|$ for $i = 1, 2, \ldots, n$ and $m = \sum_{i=1}^{n} \text{frac}(\epsilon_i)$, then m the *data-inherent error* is equal to m.

Using again the previous examples, the seat allocation method HA negates the principle of proportional representation on $V = \{5700, 2700, 1200, 400\}$ with 1 degree of negation. The method-induced error is 1 and the data-induced error is $0.15 + 0.35 + 0.6 + 0.2 = 1.3$.
The LR method affirms the principle of proportional representation on $V = \{5700, 2700, 1200, 400\}$ where the data-induced error is $0.15 + 0.35 + 0.4 + 0.2 = 1.1$.

Theorem 2
Let $V = \{v_1, v_2, \ldots, v_n\}$ be the set of votes in a party-list election. Then the LR method affirms the principle of proportional representation on V.

The seat allocation method LR affirms the principle of proportional representation on V, if $|\text{error}_{LR(v)}| < 1$ for all $v \in V$.
In the LR method, the absolute value of the seat allocation error on $v \in V$ is

$$\begin{aligned} |\ \text{error}_{LR(v)}\ | &= \left| \frac{v s_T}{v_T} - \text{int}\left(\frac{v s_T}{v_T}\right) - \delta \right| \\ &= \left| \text{frac}\left(\frac{v s_T}{v_T}\right) - \delta \right| \end{aligned}$$

where $\delta = 0$ or $\delta = 1$. Since $\text{frac}\left(\dfrac{v s_T}{v_T}\right) < 1$, it follows that $|\ \text{error}_{LR(v)}\ | < 1$. □

Theorem 3
Let $V = \{v_1, v_2, \ldots, v_n\}$ be the set of votes in a party-list election such that $v_T = v_1 + v_2 + \ldots + v_n$ with $v_1 \geq v_2 \geq \cdots \geq v_n$ and s_T be the total number of seats available for the party-list.
Suppose that $m_i = \text{int}\left(\dfrac{v_i s_T}{v_T}\right)$ for $v_i \in V$. Then $HA(v_i) \geq m_i$.

Proof.
Since $v_1 \geq v_2 \geq \cdots \geq v_n$, it follows that $m_1 \geq m_2 \geq \cdots \geq m_n$. Suppose that for some k, $m_k \geq 1$ and $m_{k+1} = 0$ where $k \in \{1, 2, \ldots, n\}$. Then $m_{k+2} = \cdots = m_n = 0$.

(a) We show that $\dfrac{v_j}{m_j} > \dfrac{v_i}{m_i + 1}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$.

Since $m_i \leq \dfrac{v_i s_T}{v_T} < m_i + 1$, we have $\dfrac{v_i s_T}{v_T(m_i + 1)} < 1$.

Since $1 \leq \dfrac{v_j s_T}{m_j v_T}$ where $j = 1, 2, \ldots, k$, it follows that

$$\frac{v_i s_T}{v_T(m_i + 1)} < \frac{v_j s_T}{m_j v_T}.$$

Hence, $\dfrac{v_i}{m_i + 1} < \dfrac{v_j}{m_j}$ where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$.

(b) We show that $\dfrac{v_j}{m_j} > v_i$ where $j = 1, 2, \ldots, k$ and $i = k + 1, \ldots, n$.

Since $0 \leq \dfrac{v_{k+1}s_T}{v_T} < 1$ and $1 \leq \dfrac{v_j s_T}{m_j v_T}$, we have $\dfrac{v_{k+1}s_T}{v_T} < \dfrac{v_j s_T}{m_j v_T}$. Thus, for $j = 1, 2, \ldots, k$,

$$\frac{v_j}{m_j} > v_{k+1}.$$

Since $v_{k+1} \geq \ldots \geq v_n$, we have for $i = k+1, \ldots, n$ and $j = 1, 2, \ldots, k$,

$$\frac{v_j}{m_j} > v_i.$$

(c). We show that $m_1 + m_2 + \ldots + m_k \leq s_T$.

Since $m_i = 0$ for $i = k+1, \cdots, n$ and $m_i \leq \dfrac{v_i s_T}{v_T} < m_i + 1$, we have

$$\begin{aligned} m_1 + m_2 + \cdots + m_k &= m_1 + m_2 + \cdots + m_n \\ &\leq \frac{v_1 s_T}{v_T} + \frac{v_2 s_T}{v_T} + \cdots + \frac{v_n s_T}{v_T} \\ &\leq \frac{(v_1 + v_2 + \ldots + v_n)s_T}{v_T} \\ &\leq s_T \end{aligned}$$

Hence, by **(a)**, **(b)** and **(c)**, each of the quotients

$$\frac{v_i}{1}, \frac{v_i}{2}, \cdots, \frac{v_i}{m_i}$$

implies a seat for party p_i, for $i = 1, 2, \ldots, k$.

Therefore, $HA(v_i) \geq m_i$. □

Definition 4

Let S be a set of real numbers and $a \in S$. The m-rank of a in S denoted by m-rank$(a; S)$ is equal to the number of elements in S that are greater than or equal to a.

Theorem 4

Let HA be the seat allocation method applied on V in a party-list election. Suppose that

$$Q = \left\{ \frac{v_i}{j} \middle| \, i = 1, 2, \ldots, n; j \in \mathbb{Z}^+ \right\}$$

where n is the number of elements in V.

Then m-rank $\left(\dfrac{v_i}{j}; Q\right) = \displaystyle\sum_{k=1}^{n} \text{int}\left(\frac{j * v_k}{v_i}\right)$.

Proof. Suppose that $m_t = \text{int}\left(\dfrac{j * v_t}{v_i}\right)$ where $m_t \leq s_T$. Then $m_t = \text{int}\left(\dfrac{v_t}{v_i/j}\right)$.

Thus, $m_t \leq \dfrac{v_t}{v_i/j} < m_t + 1$. Hence, $\dfrac{v_i}{j} \leq \dfrac{v_t}{m_t}$ if $m_t \neq 0$ and $\dfrac{v_i}{j} > \dfrac{v_t}{m_t + 1}$.

If $m_t \geq 1$, then for $k = 1, 2, \ldots, m_t$,

$$\frac{v_t}{k} \geq \frac{v_i}{j}. \qquad (2)$$

Hence, there are m_t elements in Q that are greater than or equal to v_i/j. These elements are

$$v_t/1, v_t/2, \ldots, v_t/m_t$$

where $m_t = \text{int}\left(\dfrac{v_t}{v_i/j}\right)$.

Since $v_i/1, v_i/2, \ldots, v_i/(j-1)$ are greater than v_i/j, it follows that the m-rank of v_i/j is

$$m_1 + m_2 + \ldots + m_{t-1} + m_t + m_{t+1} + \ldots + m_n$$

where $m_t = \text{int}\left(\dfrac{v_t}{v_i/j}\right)$, for $t = 1, 2, \ldots, n$.

Therefore, m-rank $\left(\dfrac{v_i}{j}; Q\right) = \displaystyle\sum_{k=1}^{n} \text{int}\left(\frac{j * v_k}{v_i}\right)$. □

Theorem 5

Let V be the set of votes in a party-list election where s_T is the available number of seats for the party-list and v_T is the total number of votes for the party-list. If $\displaystyle\sum_{k=1}^{n} \text{int}\left(\frac{(m_i + \delta) * v_k}{v_i}\right) \leq s_T$ where $\delta \geq 2$ and $m_i = \text{int}\left(\dfrac{v_i s_T}{v_T}\right)$ then the HA method negates the principle of proportional representation on v_i. The degree of negation on v_i is at least $\delta - 1$.

Proof.

If $\displaystyle\sum_{k=1}^{n} \text{int}\left(\frac{(m_i + \delta) * v_k}{v_i}\right) \leq s_T$ where $\delta \geq 2$ and $m_i = \text{int}\left(\dfrac{v_i s_T}{v_T}\right)$, then $H(v_i) \geq m_i + \delta$ where $\delta \geq 2$.

$$\begin{gathered} m_i \leq \frac{v_i s_T}{v_T} < m_i + 1 \\ m_i - (m_i + \delta) \leq \frac{v_i s_T}{v_T} - H(v_i) < m_i + 1 - (m_i + \delta) \\ -\delta \leq \frac{v_i s_T}{v_T} - H(v_i) < -\delta + 1 \\ \delta - 1 < \left| \frac{v_i s_T}{v_T} - H(v_i) \right| \leq \delta \end{gathered}$$

Since $\delta \geq 2$, we have $1 \leq \delta - 1 < \left|\text{error}_{H(v_i)}\right| \leq \delta$. This means that HA negates the principle of proportional representation on $v \in V$ and the degree of negation is at least $\delta - 1$.

3 2004 Philippine Party-List Election

The Republic Act 7941 known as Philippine Party-List System Act declares that the party-list system is a mechanism of proportional representation in the election of members to the House of Representatives. It reiterates the mandate of the 1987 Philippine Constitution that 20% of the total seats of the House of Representatives shall come from the party-list. This means that there is 1 party-list representative for every 4 representatives from the congressional districts.

In the 2004-2007 Congress, there are 212 congressional districts. Hence, the constitution mandates that there will be $\frac{212}{4} = 53$ seats for the party-list system.

RA 7941 specifies that a registered party in the party-list system which obtains 2% of the total votes for the party-list is entitled to one seat and that no party shall receive more than 3 seats. Hence, the Commission on Election (COMELEC) allocates

- one seat to a party that obtains at least 2% but less than 4% of the total votes for the party-list,
- two seats to a party that obtains at least 4% but less than 6% of the total votes for the party-list, and
- three seats to a party that obtains at least 6% of the total votes.

In the 2004 party-list election, the total number of votes for the party-list system is 12,721,952. Out of the 66 registered parties only 16 received at least 2% of the total votes and the COMELEC distributed 24 seats only. See Table 3.

We find the seat allocation error of the COMELEC allocation of seats on the result of the 2004 party-list election. See Table 3.
The COMELEC'S allocation of seats results in the negation of the principle of proportional representation on the votes of $BAYANMUNA$, $CIBAC$, ABA-AKO and $ANAD$.

The negation on the votes of $BAYAN - MUNA$ is due to the 3-seat ceiling and the 2% minimum vote threshold. The minimum vote threshold shall be $\frac{1}{53}$ or 1.886792%.

Table 3: 2004 Partylist Election

	p_i	v_i	% share of votes	No. of Seats
1	BAYAN MUNA	$1,203,305$	9.45849%	3
2	APEC	$934,995$	7.34946%	3
3	AKBAYAN!	$852,473$	6.70080%	3
4	BUHAY	$705,730$	5.54734%	2
5	AP	$538,396$	4.23202%	2
6	CIBAC	$495,193$	3.89243%	1
7	GABRIELA	$464,586$	3.65185%	1
8	PM	$448,072$	3.52204%	1
9	BUTIL	$429,259$	3.37416%	1
10	AVE	$343,498$	2.70004%	1
11	ALAGAD	$340,977$	2.68023%	1
12	VFP	$340,759$	2.67851%	1
13	COOP-NATCCO	$270,950$	2.12978%	1
14	AMIN	$269,750$	2.12035%	1
15	ALIF	$269,345$	2.11717%	1
16	AN WARAY	$268,164$	2.10788%	1
17	ABA-AKO	$251,597$	1.97766%	0
18	ANAD	$244,137$	1.91902%	0
				24

The negation on the votes of $CIBAC$, ABA-AKO, and $ANAD$ is due to the 2% minimum vote threshold.

The COMELEC's allocation of seats negates the principle of proportional representation on the set of votes V of the 2004 party-list election where the degree of negation is equal to 5 seats.

Hence, the need to amend RA 7941.

4 CONCLUSION

The largest remainder method is best method with respect to affirming the principle of proportional representation.
The highest average method negates the principle of proportional representation in some instances where the number of seats allocated is more than what is due to a party.
The seat allocation method specified in Republic Act 7941 must be revised as it results in the negation of the principle of proportional representation because of the imposition of the 3-seat cap and the 2% minimum vote threshold. We propose that the largest remainder method shall be used.

Table 4: COMELEC's Seat Allocation Error

	p_i	$\frac{v_i s_T}{v_T}$	No. of Seats	seat alloc. error
1	BAYAN MUNA	5.01300	3	2.01300
2	APEC	3.89521	3	0.89521
3	AKBAYAN!	3.55143	3	0.55143
4	BUHAY	2.94009	2	0.94009
5	AP	2.24297	2	0.24297
6	CIBAC	2.06299	1	1.06299
7	GABRIELA	1.93548	1	0.93548
8	PM	1.86668	1	0.86668
9	BUTIL	1.78830	1	0.78830
10	AVE	1.43102	1	0.43102
11	ALAGAD	1.42052	1	0.42052
12	VFP	1.41961	1	0.41961
13	COOP-NATCCO	1.12879	1	0.12879
14	AMIN	1.12379	1	0.12379
15	ALIF	1.12210	1	0.12210
16	AN WARAY	1.11718	1	0.11718
17	ABA-AKO	1.04816	0	1.04816
18	ANAD	1.01708	0	1.01708

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