On integer sequences induced by the Collatz Problem

Henry N. Adorna
Department of Computer Science, UPDiliman
Velasquez Ave., Diliman
Quezon City 1101
henri@csp.org.ph

Alvin Niño Aragon
Institute of Mathematics, UPDiliman
C.P. Garcia Ave., Diliman
Quezon City 1101
alvin.aragon@up.edu.ph

Paul John A. Cabral
Department of Computer Science
UP College Cebu
Cebu City
paul.cabral@up.edu.ph

ABSTRACT

We consider the integers in the famous Collatz Problem, classified according to their trajectories. We look at the distribution of cardinality of S_i for $1 \leq i \leq 199$, and the distribution of ratio of primes over odd elements of S_i, where S_i is the set of integers n whose trajectories contains i instances of the map $n \rightarrow 3n+1$, such that $0 \leq n \leq 100,000,000$. The behavior of the graph is similar. We also investigate the odd integer sequence in the trajectory of positive integer n, denoted by $OT(n)$. We realize that for all $i \geq 2$, we can find an $n \in S_i$, such that there exists a decreasing subsequence of $OT(n)$. We also define new ternary infinite sequence base on S_i. We show that this sequence is 2-automatic and therefore is an image under coding of a uniform morphism.

1. INTRODUCTION

Let f be a function on the set of natural numbers defined as follows:

$$f(n) = \begin{cases} \frac{n}{2} & \text{, if } n \text{ is even} \\ 3n+1 & \text{, if } n \text{ is odd} \end{cases}$$

We denote by $f^i(n)$ the i iterate of the function f evaluated at n, that is,

$$f^i(n) = \overbrace{f(f(f(f \cdots f}^{\text{i times}}(n) \cdots)))$$

The sequence of iterates

$$(n, f(n), f^2(n), f^3(n), \ldots)$$

is called the **trajectory of** n. In [5], the authors classified the possible trajectories based on Lagarias [3] as follows:

convergent: $\exists\, i \geq 0$ s.t. $f^i(n) = 1$;

non-trivial cyclic: *the sequence* $f^i(n)$ *is eventually periodic, and* $f^i(n) \neq 1$ *for all* $i \geq 0$;

divergent: $\lim_{i \to \infty} f^i(n) = \infty$.

The Collatz Problem asks for all natural numbers n does there exist a natural number i such that $f^i(n) = 1$? This means asking if every trajectory is convergent [5]. This problem is still unsolved since at least 1952. However, Feinstein showed that this problem is unprovable [1].

Let the trajectory of n be convergent. Let $g(n)$ be the number of odd numbers in the trajectory of n. In particular, let

$$g(n) = \begin{cases} 0 & \text{, if } n = 1 \\ g(\frac{n}{2}) & \text{, if } n \text{ is even} \\ g(3n+1)+1 & \text{, if } n > 1 \text{ is odd} \end{cases}$$

Otherwise, we define $g(n) = \infty$.

Example 1. *Let* $n = 2$. *Since* n *is even, we have* $g(\frac{2}{2}) = g(1) = 0$.

EXAMPLE 2. *Let $n = 3$. Since n is odd, we have*

$$\begin{aligned} g(3(3)+1)+1 &= g(10)+1 \\ &= g(5)+1 \\ &= g(3(5)+1)+1 \\ &= g(16)+1 \\ &= g(8)+1 \\ &= g(4)+1 \\ &= g(2)+1 \\ &= 1+1 \\ &= 2 \end{aligned}$$

Realized that $g(2) = 1$ in Example 1.

Using the algorithm below we can generate values of $g(n)$ for all natural number n efficiently.

AN ALGORITHM FOR GENERATING VALUES OF $g(n)$

```
T[n]← -1, for all n

T[1]← 0

G(n) begin

if T[n] is not -1 then
                    return T[n]

if n is even then
                   T[n]← G(n/2)
                    return T[n]

else
                 T[n]← G(3n+1)+1
                    return T[n]

end
```

The above algorithm uses dynamic table in the computation of the values of $g(n)$. Initially the table T has values -1 for all its entries to mean `T[n]` has not yet been determined. If $g(n)$ is computed, the result (a value which is not -1) is saved at `T[n]`. And in the succeeding computation the table is consulted and/or updated.

We provide in Table 1 below values of $g(n)$ for $1 \leq n \leq 199$ using the algorithm above. To realize the value, say, of $g(167)$ in Table 1, we look at row 16 and column 7. Hence, $g(167) = 23$.

Shallit et al. [5] defined sets of integers based on their trajectories. In particular, they define the set $S_i = \{n \geq 1 | g(n) = i\}$ as the set of integers n whose trajectories (until such that $f^k(n) = 1$, for some integer k,) contains i instances of the map $n \rightarrow 3n + 1$.

EXAMPLE 3. *Referring to Table 1 it is easy to realize the following set:*

$$S_0 = \{ 1, 2, 4, 8, 16, 32, 64, \dots \}$$

$$S_1 = \{ 5, 10, 20, 21, 40, 41, \dots \}$$

$$S_2 = \{ 3, 6, 12, 13, 24, 26, \dots \}$$

	0	1	2	3	4	5	6	7	8	9
0		0	0	2	0	1	2	5	0	6
1	1	4	2	2	5	5	0	3	6	6
2	1	1	4	4	2	7	2	41	5	5
3	5	39	0	8	3	3	6	6	6	11
4	1	40	1	9	4	4	4	4	2	7
5	7	7	2	2	41	41	5	10	5	10
6	5	5	39	39	0	8	8	8	3	3
7	3	37	6	42	6	3	6	6	11	11
8	1	6	40	40	1	1	9	9	4	9
9	4	33	4	4	38	38	2	43	7	7
10	7	7	7	31	2	12	2	36	41	41
11	41	24	5	2	10	10	5	5	10	10
12	5	34	5	15	39	39	39	15	0	44
13	8	8	8	8	8	13	3	32	3	13
14	3	3	37	37	6	42	42	42	6	6
15	3	3	6	11	6	30	11	11	11	18
16	1	35	6	6	40	40	40	23	1	16
17	1	45	9	9	9	28	4	9	9	9
18	4	4	33	33	4	14	4	14	38	38
19	38	14	2	43	43	43	7	7	7	43

Table 1: Values of $g(n)$ for $1 \leq n \leq 199$

A partial list of elements of S_i for some i can be obtained from Table 1. Note that if the trajectory of an integer n does not converge to 1, then $n \in S_\infty$. In [5], the authors showed that each S_i is infinite.

2. DISTRIBUTION OF INTEGERS IN S_I

Let us have a look at the distribution of the cardinality of S_i for some $i \geq 0$ in some subset P of positive integers. We take a subset P such that $1 \leq |P| \leq 100,000,000$ and observe the distribution of the cardinality of S_i for $0 \leq i \leq 199$. Interestingly, the cardinality $|S_i|$ for some $0 \leq i \leq 199$ will reach a peak and goes down until it reaches nullity.

In particular, in our case for $29 < i < 80$ the value of $|S_i|$ ranges between 1000000 to 2000000 and S_i reaches its peak at $i = 62$, where $|S_{62}| = 1,764,061$. If $20 < i < 29$, and $80 < i < 90$, we have $1000000 < |S_i| < 500000$. If $0 < i < 20$ and $90 < i \geq 199$, $|S_i|$ is between 0 and 500000. The graph of the distribution is given at Appendix A

We also consider the distribution of the number of odd integers and primes in S_i. It seems that at P such that $1 \leq |P| \leq 100,000,000$ and S_i for $0 \leq i \leq 199$, the distribution follows the same graph as above. The ratio between the number of primes over the number of odd integers in S_i is depicted in the following graph at Appendix B. The ratio oscillates at 11 percent, with extreme value at S_2 : of its 523 elements, 57 are odd and 18 of which (31 percent) are prime numbers.

Note that there are investigations on the distribution of prime numbers in the Collatz sequence. In this note we consider primes in each S_i.

3. ODD INTEGERS IN THE TRAJECTORY

Let n be any positive integer. We denote by $T(n)$ the trajectory of n. Realize that $T(n)$ forms the sequence

$$(n, f(n), f^2(n), f^3(n), \ldots)$$

Then we consider the following subsequence $(a_k)_{k\geq 1}$ of odd integers of $T(n)$. Let us call it **odd trajectory of** n and denote by $OT(n)$.

LEMMA 1. *Let $T(n)$ be the trajectory of a positive integer n. Let a_k be the k^{th} odd integer in the subsequence $OT(n)$ of $T(n)$. Then*

$$3a_k + 1 = 2^{\theta_k} a_{k-1},$$

for some positive integer $\theta_k \geq 1$.

PROOF: Let n be an integer and consider $T(n)$. Let a_{k-1} be the next odd integer in $T(n)$ after a_k, for some integer k. Then $a_k, x_1, x_2, \cdots x_r, a_{k-1}$ is a subsequence in $T(n)$, where the x_i's are even integers. Since a_k is an odd integer, then $x_1 = 3a_k + 1$. Then $x_i = \frac{x_{i-1}}{2}$, for all $2 \leq i \leq r$ will be the sequence of even integers between a_k and a_{k-1} in $T(n)$. Finally, $a_{k-1} = \frac{x_1}{2^r}$. Thus, let $r = \theta_k$, then

$$3a_k + 1 = 2^{\theta_k} a_{k-1}$$

as desired. □

LEMMA 2. *Let a_k and a_{k-1} be consecutive odd integer in the odd trajectory $OT(n)$ of a positive integer n. Then*

$$a_k = a_{k-1} \text{ if and only if } a_k = 1.$$

PROOF: It is easy to see that if $a_k = a_{k-1}$, we will have $3a_k+1 = 2^{\theta_k} a_k$. Then $2^{\theta_k} - 3 = 1$ and $a_k = 1$ or $2^{\theta_k} - 3 = -1$ and $a_k = -1$. Since $a_k > 0$, therefore $a_k = 1$.

The other direction is trivial. □

LEMMA 3. *Let a_k and a_{k-1} be consecutive odd integer in the odd trajectory $OT(n)$ of a positive integer n.*

If $a_k < a_{k-1}$ then $3a_k + 1 = 2a_{k-1}$.

PROOF: By hypothesis $a_k < a_{k-1}$, then $a_{k-1} = a_k + r$, for some $r > 0$. By Lemma 1, we have $3a_k + 1 = 2^{\theta_k}(a_k + r) = 2^{\theta_k} a_k + 2^{\theta_k} r$. Thus, $(3 - 2^{\theta_k})a_k = 2^{\theta_k} r - 1$. Since $r > 0, \theta_k \geq 1$, then $2^{\theta_k} r - 1 > 0$.

If $\theta_k > 1$, we have a contradiction on the value of $(3 - 2^{\theta_k})a_k$. Therefore, $\theta_k = 1$. □

Let $n \in S_i$, for some $i \geq 0$. Then every $n \in S_i$ defines a trajectory that has i odd integers which excludes 1.

LEMMA 4. *Let $n \in S_i$, for some $i \geq 0$. Then there exists $m \in S_i$ such that $OT(n) = OT(m)$.*

PROOF: Let $n \in S_i$ for some $i \geq 0$ be an odd integer. Let $m = 2^r n$, for some positive integer r. By Lemma 1 the next positive integer in the sequence will be n, then the conclusion follows.

If $n \in S_i$ is a positive even integer, then for some positive integer r, let $x = \frac{n}{2^r}$ be the first odd integer in $OT(n)$. Let $m = 2^k x$, for some positive integer k, then the desired conclusion follows. □

LEMMA 5. *Let $n \in S_i$, for some $i \geq 0$. Then there exists $m \in S_{i+1}$ such that $OT(n)$ is a subsequence of $OT(m)$.*

PROOF : Without loss of generality, let $n \in S_i$ for some $i \geq 0$ be an odd integer. Let m be an odd integer such that $m = \frac{2^r n - 1}{3}$, for some positive integer r. It is not hard to see that m is not in S_i but in $S_{i=1}$.

If m is an even integer, then let $m = 2^k(\frac{2^r n - 1}{3})$, for some positive integers k and r. It is easy to see that $m \in S_{i+1}$. □

THEOREM 1. *For all $i \geq 2$, there exists an $n \in S_i$, such that there exists a decreasing subsequence of $OT(n)$.*

PROOF: Without loss of generality, let $n \in S_i$ be an odd integer and let $a_i = n$. Let k be a positive integer such that $1 \leq k \leq i$. Let $(a_{i_j})_{0 \leq j \leq r}$, where $i_0 = k$ and $0 \leq i_r < k$. We claim that:

$$a_{i_k} < a_{i_{k-1}} \iff a_{i_k} = 4z + 3, \text{ for some integer } z.$$

We now prove the claim. By Lemma 3, we have

$$3a_{i_k} + 1 = 2a_{i_{k-1}}. \tag{1}$$

Since a_{i_k} and $a_{i_{k-1}}$ are odd integers, then let $a_{i_k} = 2m+1$, for some integer m and let $a_{i_{k-1}} = 2q + 1$, for some integer q. By substituting these values in Equation 1, one obtains

$$m = \frac{2q - 1}{3}, \tag{2}$$

which implies that 3 divides $2q - 1$, since m is an integer. Thus $q \cong 2(\mod 3)$, then $q = 3t + 2$, for some integer t. Then Equation 2 yields that $m = 2t + 1$. Therefore, $a_{i_k} = 2(2t + 1) + 1 = 4t + 3$ as desired.

For the other direction, let $a_{i_k} = 4z + 3$, for some integer z. Then by Equation 1, we have $3(4z + 3) + 1 = 12z + 10 = 2(6z + 5)$. Realize that $6z + 5$ is odd and $6z + 5 > 4z + 3$. □

4. INFINITE SEQUENCE BASED ON S_I

We will construct a new infinite ternary sequence based on the trajectories of integers. First, let us fixed our terminology. For detailed discussions on automatic sequences we refer to [4] and for automata and formal languages we refer to [2] and [6].

We will call a finite set Σ an *alphabet* and its elements *symbols* or *letters*. Concatenation of symbols in Σ will be called *words* or *strings*. By Σ^+, we mean set of all possible non-empty strings over Σ.

A subset L of Σ^* is called a *language*. A *regular expression* over the alphabet Σ is a well formed word over the alphabet $\Sigma \cup \{\lambda, \emptyset, (,), +, \star\}$. If the word u is a regular expression, then $L(u)$ represents a language that u specifies. A language L is regular if $L = L(u)$ for some regular expression u. In particular, every finite language is called *regular.*

We are aware of the standard finite automaton models which either accept or reject any given input string [2], [6]. We will be interested in more general models of function computation by finite automata. Let w be an input string. The automaton moves from state to state according to its transition function δ, while reading the symbols in w. After reading the whole string w, the automaton halts in a state, say q. Then the automaton outputs $\tau(q)$, where τ is the output mapping. This automaton is called an automaton with output. In particular, we define a *deterministic finite automaton with output* (DFAO) as 6-tuple $M = (Q, \Sigma, \delta, q_0, \Delta, \tau)$ where the sets Q, Σ, δ and q_0 are define classically as in a DFA in [2] and [6], Δ is the output alphabet and $\tau : Q \longrightarrow \Delta$ is the output function. Machine M defines a function from Σ^* to Δ, which we denote as $f_M(w)$, as follows:

$$f_M(w) = \tau(\delta(q_0, w)).$$

In this note we will be particularly interested in the case where the input represents a number in base k, for some positive integer $k \geq 2$. If this is the case our DFAO will be called k-DFAO.

Definition 1. *The sequence $(a_n)_{n\geq 0}$ over a finite alphabet Δ is called k-automatic if there exists a k-DFAO $M = (Q, \Sigma, \delta, q_0, \Delta, \tau)$ such that $a_n = \tau(\delta(q_0, w))$ for all $n \geq 0$ and all w with $[w]_k = n$.*

Note that this definition requires that the automaton returns the correct answer even if the input possesses leading zeros.

Alternatively, we can characterize automatic sequences as follows. Let $(a_n)_{n\geq 0}$ be a sequence over Δ, let $k \geq 2$ be an integer and let $d \in \Delta$. Define the set $I_k((a_n)_{n\geq 0}, d) = \{[n]_k \mid a_n = d\}$ as *k-fiber.*

Lemma 6. *[4] The sequence $(a_n)_{n\geq 0}$ is k-automatic iff each of the fibers $I_k((a_n)_{n\geq 0}, d)$ is a regular language for all $d \in \Delta$.*

Let us now define a sequence $(c_n)_{n\geq 0}$ as follows:

$$c_n = \begin{cases} 0 & \text{, if } n = 0 \\ g(n) & \text{, if } g(n) = 1 \text{ or } g(n) = 0 \\ 2 & \text{, otherwise} \end{cases}$$

Hence

$$(c_n)_{n\geq 0} = 00020122021222220222112222222220 \cdots$$

We list in Table 2 below the first 200 terms of $(c_n)_{n\geq 0}$. To know the term c_{168} in the sequence, we look at row 16 and column 8, which gives a 1.

	0	1	2	3	4	5	6	7	8	9
0	0	0	0	2	0	1	2	2	0	2
1	1	2	2	2	2	2	0	2	2	2
2	1	1	2	2	2	2	2	2	2	2
3	2	2	0	2	2	2	2	2	2	2
4	1	2	1	2	2	2	2	2	2	2
5	2	2	2	2	2	2	2	2	2	2
6	2	2	2	2	0	2	2	2	2	2
7	2	2	2	2	2	2	2	2	2	2
8	1	2	2	2	1	1	2	2	2	2
9	2	2	2	2	2	2	2	2	2	2
10	2	2	2	2	2	2	2	2	2	2
11	2	2	2	2	2	2	2	2	2	2
12	2	2	2	2	2	2	2	2	0	2
13	2	2	2	2	2	2	2	2	2	2
14	2	2	2	2	2	2	2	2	2	2
15	2	2	2	2	2	2	2	2	2	2
16	1	2	2	2	2	2	2	2	1	2
17	1	2	2	2	2	2	2	2	2	2
18	2	2	2	2	2	2	2	2	2	2
19	2	2	2	2	2	2	2	2	2	2

Table 2: The first 200 terms of $(c_n)_{n\geq 0}$.

We redefine S_i as follows

$$S_i = \{\, [n]_2 \mid g(n) = i \,,\, n \geq 0 \,\}.$$

Here we have S_i as the set that contains the binary representation of a positive integer n with $g(n) = i$.

Lemma 7. *The fibers of $(c_n)_{n\geq 0}$ are all regular languages.*

Proof: It is not hard to see that the following are the fibers of $(c_n)_{n\geq 0}$ over $\{0, 1\}$;

$$I_2((c_n)_{n\geq 0}, 0) = S_0 := L(10^*)$$

$$I_2((c_n)_{n\geq 0}, 1) = S_1 := L(1(01)^*010^*)$$

$$I_2((c_n)_{n\geq 0}, 2) := \{0, 1\}^* - (S_0 \cup S_1)$$

Therefore, they are all regular languages. □

Note that Shallit et al. [5] proved that S_i is 2-automatic for all $i \geq 0$. The following Theorem follows directly from Lemma 6 and Lemma 7.

Theorem 2. *$(c_n)_{n\geq 0}$ is 2-automatic.*

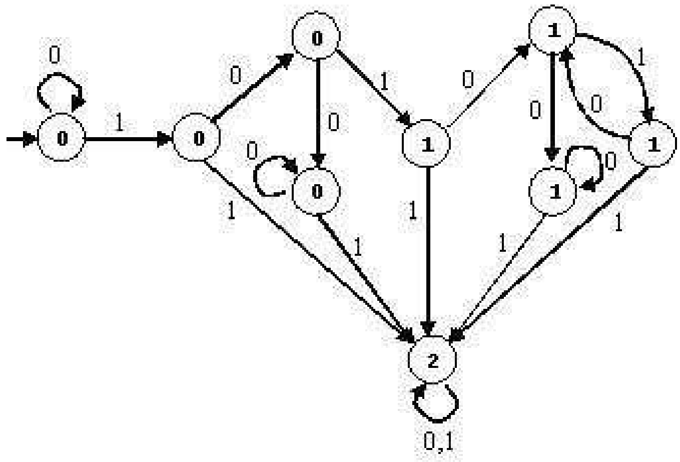
Moreover we can construct a 2-DFAO

$$M = (Q, \Sigma, \delta, q_0, \Delta, \tau),$$

for $(c_n)_{n\geq 0}$, where

$$Q = \{A, B, C, D, E, F, G, H, I\},$$

$$\Sigma = \{0, 1\}, q_0 = A, \Delta = \{0, 1, 2\}.$$



The transition function δ is defined as follows:

δ	0	1
A	A	B
B	C	I
C	D	E
D	D	I
E	F	I
F	G	H
G	G	I
H	F	I
I	I	I

and $\tau \colon Q \longrightarrow \Delta$ is defined as follows

$$\tau(A) = \tau(B) = \tau(C) = \tau(D) = 0$$

$$\tau(E) = \tau(F) = \tau(G) = \tau(H) = 1$$

$$\tau(I) = 2$$

To see that M is indeed a 2-DFAO for $(c_n)_{n\geq 0}$, it is enough to verify

$$\tau(\delta(A, w)) = \begin{cases} 0 & \text{for } w \in S_0, \\ 1 & \text{for } w \in S_1, \\ 2 & \text{otherwise.} \end{cases}$$

5. IMAGE UNDER CODING OF A UNIFORM MORPHISM

We will show that $(c_n)_{n\geq 0}$ is an image under coding of a uniform morphism at a fixed point. The existence of these morphisms is assured by Cobham's Theorem. First, we define the following;

DEFINITION 2. *A* **morphism** *is a map φ from Σ^* to Δ^* satisfying $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in \Sigma^*$.*

If for all $x \in \Sigma$, there is positive integer k such that $|\varphi(x)| = k$, then we call φ a **k-uniform morphism.**

In particular, if for all $x \in \Sigma$, there is positive integer k such that $|\varphi(x)| = 1$, φ is called a **coding.**

Let $a \in \Sigma$. If $\varphi(ax) = ax$, for some $x \in \Sigma^*$ with $|x| = k - 1$, we say that φ is **prolongable** on a. In this case, the infinite word

$$\mathbf{w} = \varphi^{\omega}(a) = ax\varphi(x)\varphi^2(x)\varphi^3(x)\cdots$$

is the unique fixed point of φ starting with a.

Let us consider the following 2-uniform morphism φ, based on the transitions in the 2-DFAO M for $(c_n)_{n\geq 0}$.

$$\begin{array}{lll} \varphi(A) = AB & \varphi(D) = DI & \varphi(G) = GI \\ \varphi(B) = CI & \varphi(E) = FI & \varphi(H) = FI \\ \varphi(C) = DE & \varphi(F) = GH & \varphi(I) = II \end{array}$$

where $\Sigma = \{A, B, C, D, E, F, G, H, I\} = \Delta$.

Let

$$\begin{aligned} X = \quad \varphi^{\omega}(A) &= AB\varphi(B)\varphi^2(B)\varphi^3(B)\varphi^4(B)\cdots \\ &= ABCI\varphi^2(B)\varphi^3(B)\varphi^4(B)\cdots \\ &= ABCIDEII\varphi^3(B)\varphi^4(B)\cdots \\ &= ABCIDEIIDIFIIIII\cdots \end{aligned}$$

Then

$$\begin{aligned} \varphi(X) &= \varphi(ABCIDEIIDIFIIIII\cdots) \\ &= ABCIDEIIDIFIIIII\cdots \\ &= X \end{aligned}$$

Certainly, $\varphi(X) = X$ implies that X is the unique infinite fixed point of φ starting with A.

Now, let us use the morphism $\tau \colon Q \longrightarrow \Delta$ is defined above as follows

$$\tau(A) = \tau(B) = \tau(C) = \tau(D) = 0$$

$$\tau(E) = \tau(F) = \tau(G) = \tau(H) = 1$$

$$\tau(I) = 2.$$

Then get the image of $\varphi(X)$ under τ, that is,

$$\begin{aligned} \tau(\varphi(X)) &= \tau(\varphi(ABCIDEIIDIFIIIII\cdots)) \\ &= \tau(ABCIDEIIDIFIIIII\cdots) \\ &= \tau(A)\tau(B)\tau(C)\tau(I)\tau(D)\tau(E)\tau(I)\tau(I) \\ &\quad \tau(D)\tau(I)\tau(F)\tau(I)\tau(I)\tau(I)\tau(I)\tau(I)\cdots \\ &= 0002012202122222\cdots \end{aligned}$$

It is not hard to see that

$$\tau(\varphi(X)) = (c_n)_{n\geq 0}$$

since the morphisms we used are essentially the transition function δ and output function τ of the automaton M for $(c_n)_{n\geq 0}$. Finally, we have shown that,

THEOREM 3. *The infinite sequence $(c_n)_{n\geq 0}$ is an image under coding τ of a 2-uniform morphism φ fixed by A.*

Acknowledgement

We thank Ivan Orosco and Nat Tabucoy for very fruitful discussions.

6. REFERENCES

[1] Feinstein, C. A.. The Collatz 3n+1 Conjecture is Unprovable. preprint available from: arXiv:math.GM/0312309 v14 22Sep2005

[2] Hopcroft, J.E., Ullman, J.D.*Introduction to Automata Theory, Languages and Computation,* Addison-Wesley, 1979.

[3] Lagarias, J.C.. The 3x+1 problem and its generalizations. *American Mathematical Monthly* **92**. 1985. pp 3-23.

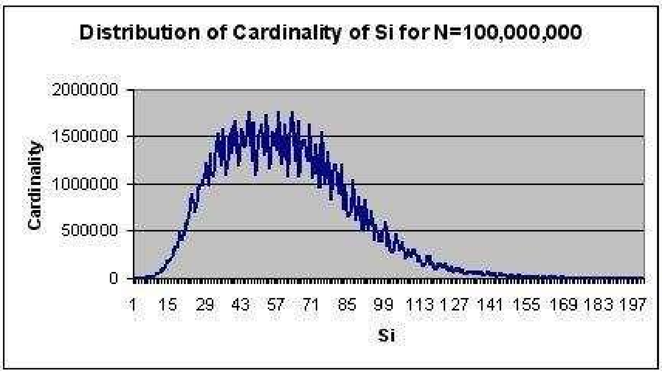
[4] Shallit, J., Allouche, J.P.. *Automatic Sequences: Theory, Applications, Generalizations.* Cambridge University Press, 2003.

[5] Shallit, J., Wilson, D.. The $3X + 1$ Problem and Finite Automata. *EATCS Bull.* 46 (1992), 182-185.

[6] Sipser, M. *Introduction to the Theory of Computation.* PWS Publishing Company, 1997.

APPENDIX

A.



B.