Infinite Language Hierarchy Based on Regular-Regulated Right-Linear Grammars with Start Strings

Alexander Meduna
Department of Information Systems
Faculty of Information Technology
Brno University of Technology
Božetěchova 2, Brno Czech Republic
meduna@fit.vutbr.cz

Lukáš Rychnovský
Department of Information Systems
Faculty of Information Technology
Brno University of Technology
Božetěchova 2, Brno Czech Republic
rychnov@fit.vutbr.cz

ABSTRACT

The present paper discusses regular-regulated right-linear grammars with start strings rather than single symbols. It demonstrates that these grammars with start strings consisting of no more than $n+1$ symbols are stronger than these grammars with start strings consisting of no more than n symbols, for all $n \geq 1$. On the other hand, these grammars with start strings of any length only generate the family of regular languages if they change the position of rewriting finitely many times during the derivation of any sentence from the generated language.

Keywords

right-linear grammars, regular regulation, infinite hierarchy of language families

1. INTRODUCTION

In this paper, we discuss right-linear grammars that start their derivations from start strings rather than single symbols. Specifically, we study these grammars regulated by regular languages. We demonstrate that the language family generated by these grammars with start strings of length n or shorter is properly included in the language family generate by these grammars with start strings of length $n+1$ or shorter, for all $n \geq 1$. From a broader perspective, by obtaining this infinite hierarchy of language families, we contribute to a classical trend of the formal language theory that demonstrates that some properties of grammars affect the language families that the grammars generate. (For more information about infinite language hierarchy see [3], [7], [4]).

Surprisingly enough, however, if during the derivation of any sentence from the generated language, these grammars change the position of rewriting finitely many times, they just generate the family of regular languages no matter how long their start strings are. In other words, only if the number of these changes is unlimited, the above hierarchy holds true.

2. PRELIMINARIES

We assume that the reader is familiar with the language theory (see [5], [1], [6]).

For a set, Q, $card(Q)$ denotes the cardinality of Q. I denotes the set of all positive integers. For an alphabet, V, V^* represents the free monoid generated by V under the operation of concatenation. The identity of V^* is denoted by ε. For a word, $w \in V^*$, $|w|$ denotes the length of w. $Sub(w)$ denotes the set of all substrings of w.

3. DEFINITIONS

In this section, we define the central notions of this paper.

DEFINITION 3.1. *Let $n \geq 1$. A* linear grammar with a start string of length n, *n-LG for short, is a quadruple $G = (N, T, R, S)$, where N and T are alphabets such that $N \cap T = \emptyset, S \in N^+, |S| \leq n$, and R is a finite set of productions of the form $A \to x$, where $A \in N$ and $x \in T^*(N \cup \{\varepsilon\})T^*$. Set $V = T \cup N$.*

Let Ψ be an alphabet of rule labels such that $card(\Psi) = card(R)$, and ψ be a bijection from R to Ψ. For simplicity, to express that ψ maps a rule $A \to x \in R$, to ρ, where $\rho \in \Psi$, we write $\rho.A \to x \in R$; in other words, $\rho.A \to x$ means $\psi(A \to x) = \rho$.

If $\rho.A \to x \in R$ and $u, v \in V^*$, then we write $uAv \Rightarrow uxv$ $[\rho]$ in G.

Let $\chi \in V^*$. Then G makes the zero-step derivation from χ to χ according to ε, symbolically written as $\chi \Rightarrow^0 \chi$ $[\varepsilon]$. Let there exist a sequence of derivation steps $\chi_0, \chi_1, \dots, \chi_n$ for some $n \geq 1$ such that $\chi_{i-1} \Rightarrow \chi_i$ $[\rho_i]$, where $\rho_i \in \Psi$, for all $i = 1, \dots, n$, then G makes n *derivation steps from χ_0 to χ_n according to $\rho_1 \dots \rho_n$, symbolically written as $\chi_0 \Rightarrow^n \chi_n$ $[\rho_1 \dots \rho_n]$. If for some $n \geq 0, \chi_0 \Rightarrow^n \chi_n$ $[\rho]$, where $\rho \in \Psi^*$ and $|\rho| = n$, we write $\chi_0 \Rightarrow^* \chi_n$ $[\rho]$.*

We call a derivation $S \Rightarrow^* w$ successful, *if and only if, $w \in T^*$.*

Let Ξ be a control language over Ψ; that is, $\Xi \in \Psi^*$.

Under the regulation by Ξ, the language that G generates is denoted by $L(G,\Xi)$ and defined as

$$L(G,\Xi) = \{w \mid S \Rightarrow^* w\ [\rho], \rho \in \Xi, w \in T^*\}.$$

Let i be a positive integer and X be a family of languages. Set

$$\mathfrak{L}(X,i) = \{L \mid L = L(G,X), \text{where } G \text{ is a } i\text{-}LG\}.$$

In the same manner we define a right-linear grammar with a start string of length n, *n-RLG for short, where R is a finite set of productions of the form $A \to x$, where $A \in N$ and $x \in T^*(N \cup \{\varepsilon\})$ and define*

$$\mathfrak{R}(X,i) = \{L \mid L = L(G,X), \text{where } G \text{ is a } i\text{-}RLG\}.$$

Specifically, $\mathfrak{R}(REG,i)$ and $\mathfrak{L}(REG,i)$ are central to this paper, where REG denotes the family of regular languages.

DEFINITION 3.2. *Let $G = (N,T,R,S)$ be an n-LG for some $n \geq 1$ (See Definition 3.1). $G = (N_1,N_2,\ldots,N_n,T,R_1,R_2,\ldots,R_n,S)$ is a* distributed n-LG, *$n{-}_{dis}LG$ for short, if*

- *$N = N_1 \cup N_2 \cup \ldots \cup N_n$, where $N_i, 1 \leq i \leq n$ are pairwise disjoint nonterminal alphabets,*
- *$S = X_1X_2\ldots X_n,\ X_i \in N_i, 1 \leq i \leq n$,*
- *$R = R_1 \cup R_2 \cup \ldots \cup R_n$,*
  such that for every $A \to xBy \in R_i,\ A,B \in N_i$, for some $1 \leq i \leq n; x,y \in T^*$
  and for every $A \to a \in R,\ A \in N, a \in T^*$.

Set $\Psi_i = \{\rho \mid \rho.A \to aBb \in R_i$ or $\rho.A \to a \in R_i$, where $A,B \in N_i$ and $a,b \in T^*\}$.

In the same manner we define a distributed n-RLG, *$n{-}_{dis}RLG$ for short, if this grammar is $n{-}_{dis}LG$ and all rules are right-linear.*

DEFINITION 3.3. *(See [2]) For $n \geq 1$, a* linear simple matrix grammar of degree *n, n-LSM for short, is an (n+3)-tuple $G = (N_1,\ldots N_n,T,S,P)$ where*

- *$N_i, 1 \leq i \leq n$ are pairwise disjoint nonterminal alphabets,*
- *T is a terminal alphabet, $N_i \cap T = \emptyset, 1 \leq i \leq n$,*
- *$S \notin N_1 \cup \ldots \cup N_n$ is the start symbol,*
- *P is a finite set of rules. P contains three kinds of rules*
  1. *$S \to x,\quad x \in T^*$,*
  2. *$S \to X_1\ldots X_n,\quad X_i \in N_i, 1 \leq i \leq n$,*
  3. *$(X_1 \to x_1, X_2 \to x_2, \ldots, X_n \to x_n)$,*
     $X_i \in N_i, x_i \in T^*N_iT^* \cup T^*, 1 \leq i \leq n$.

For $x,y \in (N \cup T \cup \{S\})^*, x \Rightarrow y$ if and only if

- *either $x = S$ and $S \to y \in P$,*
- *or $x = y_1X_1\ldots y_nX_n, y = y_1x_1\ldots y_nx_n$,*
  where $y_i \in T^*, x_i \in T^*N_iT^* \cup T^*, X_i \in N_i, 1 \leq i \leq n$ and $(X_1 \to x_1,\ldots,X_n \to x_n) \in P$.

In the same manner we define a right-linear simple matrix grammar of degree *n, n-RLSM for short, if in definition of P the last rule is*

3. *$(X_1 \to x_1, X_2 \to x_2, \ldots, X_n \to x_n)$,*
   $X_i \in N_i, x_i \in T^*N_i \cup T^*, 1 \leq i \leq n$.

For more information about simple matrix grammars, see [2].

DEFINITION 3.4. *Let $i \geq 1$ and X be a family of languages. Let $L(G,\Xi)$ be a language generated by G and regulated by Ξ (See definition 3.1). Set*

- *$\mathfrak{R}(X,i) = \{L \mid L = L(G,\Xi)$, where $G = (N,T,R,S)$ is a i-RLG and $\Xi \in X\}$.*
- *$\mathfrak{L}(X,i) = \{L \mid L = L(G,\Xi)$, where $G = (N,T,R,S)$ is a i-LG and $\Xi \in X\}$.*
- *${}_{dis}\mathfrak{R}(X,i) = \{L \mid L = L(G,\Xi)$,*
  where $G = (N_1,N_2,\ldots,N_n,T,R_1,R_2,\ldots,R_n,S)$ is a $i{-}_{dis}RLG$ and $\Xi \in X\}$.
- *${}_{dis}\mathfrak{L}(X,i) = \{L \mid L = L(G,\Xi)$,*
  where $G = (N_1,N_2,\ldots,N_n,T,R_1,R_2,\ldots,R_n,S)$ is a $i{-}_{dis}LG$ and $\Xi \in X\}$.
- *${}_{SM}\mathfrak{R}(i) = \{L \mid L = L(G)$,*
  where $G = (N_1,N_2,\ldots,N_n,T,R,S)$ is a i-RLSM$\}$.
- *${}_{SM}\mathfrak{L}(i) = \{L \mid L = L(G)$,*
  where $G = (N_1,N_2,\ldots,N_n,T,S,P)$ is a i-LSM$\}$.

4. RESULTS

LEMMA 4.1. *For every n-LG $G = (N,T,R,S)$, there exists an equivalent $n{-}_{dis}LG$ $G' = (N'_1,N'_2,\ldots,N'_n,T',R'_1,R'_2,\ldots,R'_n,S')$ such that $L(G) = L(G')$.*

PROOF. *We will define nonterminals of G' in the form (A,k) so that $(A,k) \in N'_k$. Hence,*

- *$N'_j = \{(A,j) \mid A \in N\}$, where $1 \leq j \leq n$;*
- *$T' = T$;*
- *$R'_j = \{(A,j) \to x(B,j)y \mid A \to xBy \in R$,*
  $(A,i),(B,i) \in N'_j,\ x,y \in T^*\}$ where $1 \leq j \leq n$;
- *$S' = (A_1,1)(A_2,2)\ldots(A_n,n)$, where $S = A_1A_2\ldots A_n$.*

For $G' = (N'_1, N'_2, \ldots, N'_n, T', R', S')$ holds $N'_i \cap N'_j = \emptyset$ for $i \neq j$, $1 \leq i, j \leq n$. For every derivation $a \Rightarrow b\ [\rho]$, $a, b \in \{N \cup T\}^*$, $\rho.A \rightarrow xBy \in R$, $x, y \in T^*$, $A, B \in N$ of grammar G there always exists equivalent derivation in G' in form $a' \Rightarrow b'\ [\rho']$, $a', b' \in \{N' \cup T'\}^*$, $\rho'.(A, i) \rightarrow x(B, i)y \in R'$, $x, y \in T'^*$, $(A, i), (B, i) \in N'_i$. □

LEMMA 4.2. *For every $n{-}_{dis}LG$ $G' = (N'_1, N'_2, \ldots, N'_n, T', R'_1, R'_2, \ldots, R'_n, S')$, there exists an equivalent n-LG $G = (N, T, R, S)$ such that $L(G) = L(G')$.*

PROOF. *We define grammar $G = (N, T, R, S)$ in the following way*

- $N = N'_1 \cup N'_2 \cup \ldots \cup N'_n$,
- $T = T'$,
- $R = R'_1 \cup R'_2 \cup \ldots \cup R'_n$,
- $S = A_1 A_2 \ldots A_n$, *where* $S' = A_1 A_2 \ldots A_n \in R'$.

A rigorous proof that $L(G) = L(G')$ is left to the reader. □

THEOREM 4.1. *For all $n \geq 1$, $\mathcal{L}(n{-}_{dis}LG) = \mathcal{L}(n\text{-}LG)$.*

PROOF. *This theorem directly follows from Lemma 4.1 and Lemma 4.2.* □

THEOREM 4.2. *For all $n \geq 1$, $\mathcal{L}(n{-}_{dis}RLG) = \mathcal{L}(n\text{-}RLG)$.*

PROOF. *This theorem directly follows from Theorem 4.1.* □

LEMMA 4.3. *Let $i \geq 1$. ${}_{dis}\mathfrak{L}(REG, i) \subseteq {}_{SM}\mathfrak{L}(i)$. That is, for every $n{-}_{dis}LG$ $G = (N_1, \ldots, N_n, T, R_1, \ldots, R_n, S)$ regulated by regular language Ξ there exists equivalent n-LSM $G' = (N'_1, \ldots, N'_n, T', S', P')$ such that $L(G) = L(G')$.*

PROOF. *Let $\Xi = L(G_\Xi), G_\Xi = (N_\Xi, T_\Xi, R_\Xi, S_\Xi)$. Let $R = R_1 \cup R_2 \cup \ldots \cup R_n$. We will define grammar $G' = (N'_1, \ldots, N'_n, T', S', P')$ this way:*

- $N'_1 = \{[A, X] \mid A \in N_1, X \in N_\Xi\}$,
- $N'_i = N_i, 2 \leq i \leq n$,
- $T' = T$,
- $P'_1 = \{([A_1, X], A_2, \ldots, A_n) \rightarrow (u[B_1, Y]v, A_2, \ldots, A_n) \mid$ $\mid A_i \in N_i, 1 \leq i \leq n,\ X, Y \in N_\Xi$ *and* $f.A_1 \rightarrow uB_1 v \in R_1,\ X \rightarrow fY \in R_\Xi,$ $u, v \in T^*\}$,
- $P'_2 = \{([A_1, X], A_2, \ldots, A_j, \ldots, A_n) \rightarrow$ $\rightarrow ([A_1, Y], A_2, \ldots, uB_j v, \ldots, A_n) \mid$ $\mid A_i \in N_i, 1 \leq i \leq n,\ 2 \leq j \leq n,\ X, Y \in N_\Xi$ *and* $f.A_j \rightarrow uB_j v \in R_j,\ X \rightarrow fY \in R_\Xi,$ $u, v \in T^*\}$,
- $P' = P'_1 \cup P'_2 \cup \{S' \rightarrow [X_1, S_\Xi]X_2 \ldots X_n \mid$ $\mid S = X_1 \ldots X_n \in G,\ X_i \in N_i,\ 1 \leq i \leq n\}$.

Note that P'_1 is a special case of P'_2 with $j = 1$.

Let $L_n(G) = \{x \mid S \Rightarrow^n x \text{ in } G,\ x \in \{N \cup T\}^*\}$ and $L_n(G') = \{x \mid S' \Rightarrow^{n+1} x \text{ in } G',\ x \in \{N' \cup T'\}^*\}$. We will prove that $L_n(G) = h(L_n(G'))$ for every $n \geq 0$, where h is surjective function $h : \{N'_1 \cup \ldots \cup N'_n \cup T'\} \rightarrow \{N_1 \cup \ldots \cup N_n \cup T\}$ defined as

$$h(w) = \begin{cases} A, & \text{if } w \in N'_1, w = [A, Y], \\ w, & \text{otherwise.} \end{cases}$$

First we will prove that $L_n(G) \subseteq h(L_n(G'))$ by induction on n:
Let $n = 0$.
$L_0(G) = \{X_1 X_2 \ldots X_n\}$, $L_0(G') = \{[X_1, Y]X_2 \ldots X_n\}$ because $S' \rightarrow [X_1, Y]X_2 \ldots X_n \in P'$ and, therefore, $h(L_0(G')) = \{X_1 X_2 \ldots X_n\} = L_0(G)$.
Let us suppose that the claim holds for all $n \leq k$, where k is a non-negative integer.
Let $n = k + 1$.
Consider $w \in L_{k+1}(G)$ and a derivation $S \Rightarrow^k v \Rightarrow w$ in G, so that $v \Rightarrow w\ [p]$, where $v = C_1 C_2 \ldots C_{i-1} X C_{i+1} \ldots C_n$, $w = C_1 C_2 \ldots C_{i-1} uYv C_{i+1} \ldots C_n, C_j \in N_j \cup \{T\}^*, 1 \leq j \leq n, p.X \rightarrow uYv \in R, A \rightarrow pB \in R_\Xi$. From the induction step, $v \in h(L_k(G'))$. As $([C_1, A]C_2 \ldots C_{i-1} X C_{i+1} \ldots C_n) \rightarrow ([C_1, B]C_2 \ldots C_{i-1} uYv C_{i+1} \ldots C_n) \in P'$, $w \in h(L_{k+1}(G'))$.

Now we prove that $L_n(G) \supseteq h(L_n(G'))$ by induction on $n \geq 0$:
Let $n = 0$. By analogy with the previous part of this proof.
Let us suppose that our claim holds for all $n \leq k$, where k is a non-negative integer.
Let $n = k + 1$.
Consider $w \in L_{k+1}(G')$ and a derivation $S \Rightarrow^k v \Rightarrow w$ in G', where $v = [C_1, A]C_2 \ldots C_{i-1} X C_{i+1} \ldots C_n$,
$w = [C_1, B]C_2 \ldots C_{i-1} uYv C_{i+1} \ldots C_n, C_j \in N_j \cup \{T\}^*, 1 \leq j \leq n$. From the induction step, $h(v) \in L_k(G)$. Since $p.X \rightarrow uYv \in R, A \rightarrow pB \in R_\Xi$, we have $h(w) \in L_{k+1}(G)$. □

LEMMA 4.4. *Let $i \geq 1$. ${}_{dis}\mathfrak{L}(REG, i) \supseteq {}_{SM}\mathfrak{L}(i)$ That is, for every n-LSM $G' = (N'_1, \ldots, N'_n, T', S', P')$ there exists equivalent $n{-}_{dis}LG$ $G = (N_1, \ldots, N_n, T, R_1, \ldots, R_n, S)$ regulated by regular language Ξ such that $L(G) = L(G')$.*

PROOF. *G is defined in this way*

- $N_i = N'_i, 1 \leq i \leq n$;
- $T = T'$;
- $S = S'$;
- $R_i = \{r_{ij}.A_i \rightarrow u_i B_i v_i \mid$ *for the jth rule* $(A_1, \ldots, A_i, \ldots, A_n) \rightarrow$ $\rightarrow (u_1 B_1 v_1, \ldots, u_i B_i v_i, \ldots, u_n B_n v_n) \in P'$, $u_i, v_i \in T^*,\ 1 \leq j \leq |P'|\},\ 1 \leq i \leq n$.

and $\Xi = L(G_\Xi), G_\Xi = (N_\Xi, T_\Xi, R_\Xi, S_\Xi)$ is defined

- $N_\Xi = \{Q\} \cup \{Q_{ij} \mid 1 \leq i \leq n - 1,\ 1 \leq j \leq |P'|\}$;
- $T_\Xi = \{r_{ij} \mid 1 \leq i \leq n,\ 1 \leq j \leq |P'|\}$;

- $R_\Xi = \{Q \to r_{1j}Q_{1j} \mid 1 \le j \le |P'|\} \cup$
  $\cup \{Q_{ij} \to r_{i+1j}Q_{i+1j} \mid 1 \le i \le n-2,$
  $1 \le j \le |P'|\} \cup \{Q_{n-1j} \to r_{nj}Q \mid 1 \le j \le |P'|\};$

- $S_\Xi = Q$.

□

THEOREM 4.3. *For all* $i \ge 1$, ${}_{dis}\mathfrak{L}(REG, i) = {}_{SM}\mathfrak{L}(i)$.

PROOF. *This theorem directly follows from Lemma 4.3 and Lemma 4.4* □

THEOREM 4.4. *For all* $i \ge 1$, ${}_{dis}\mathfrak{R}(REG, i) = {}_{SM}\mathfrak{R}(i)$.

PROOF. *This theorem directly follows from Theorem 4.3.* □

THEOREM 4.5. *For all* $i \ge 1$, ${}_{SM}\mathfrak{L}(i) \subset {}_{SM}\mathfrak{L}(i+1)$.

PROOF. *See [2].* □

The main results of this paper follows next.

THEOREM 4.6. *For all* $i \ge 1$,
$\mathfrak{L}(REG, i) \subset \mathfrak{L}(REG, i+1)$.

PROOF. *This theorem follows from Theorems 4.1, 4.3 and 4.5.* □

THEOREM 4.7. *For all* $i \ge 1$,
$\mathfrak{R}(REG, i) \subset \mathfrak{R}(REG, i+1)$.

PROOF. *This theorem follows from Theorem 4.6.* □

Let G be an $n-_{dis}$RLG satisfying Definition 3.2. Let $S \Rightarrow^* w\ [\sigma],\ w \in T^*, \sigma = \rho_1\rho_2\ldots\rho_m$, for some $m \ge 1, 1 \le i \le m, \rho_i \in \Psi, \sigma \in \Xi$.
Set

$$d = card(\{\rho_j\rho_{j+1} \mid j = 1, \ldots, m-1,\ \rho_j \in \Psi_k, \rho_{j+1} \in \Psi_h, k \ne h\}).$$

Then, during the generation of $w \in L(G, \Xi)$ by $S \Rightarrow^* w\ [\sigma]$, *G changes the derivation position d times.* If there is a constant $k \ge 0$ such that for every $x \in L(G, \Xi)$ there is a generation of x during which G changes the derivation position k or fewer times, then the *generation of* $L(G, \Xi)$ *by G requires no more than k changes of derivation positions.* Let k be the minimal possible than we write $d(G) = k$.

Let $i \ge 1, k \ge i-1$ and X be a family of languages. Set

- $\mathfrak{R}(X, i, k) = \{L \mid L = L(G, \Xi)$, where $G = (N, T, R, S)$ is a i-RLG, $\Xi \in X$ and $d(G) = k$, the generation of $L(G, \Xi)$ by G requires no more than k changes of derivation positions$\}$.

- ${}_{dis}\mathfrak{R}(X, i, k) = \{L \mid L = L(G, \Xi)$, where $G = (N_1, N_2, \ldots, N_n, T, R_1, R_2, \ldots, R_n, S)$ is a $i-_{dis}$RLG, $\Xi \in X$ and $d(G) = k$, the generation of $L(G, \Xi)$ by G requires no more than k changes of derivation positions$\}$.

THEOREM 4.8. *Let* $i \ge 1, k \ge 0$.
Then, $\mathfrak{R}(REG, i, k) = {}_{dis}\mathfrak{R}(REG, i, k)$.

PROOF. *This proof is analogous to the proof of Theorem 4.1.* □

DEFINITION 4.1. *Let G be an* $n-_{dis}RLG$ $G = (N_1, N_2, \ldots, N_n, T, R_1, R_2, \ldots, R_n, S)$ *regulated by regular language* Ξ. *Let* $\Xi = L(H), H = ({}_HN, {}_HT, {}_HS, {}_HP)$. *Let* $A, B \in {}_HN$.

We write $A\ {}_i\Rightarrow B$ *and say* B is achievable from A in i-th component of G in one derivation step *if and only if there exists derivation* $A \Rightarrow xB,\ x \in {}_HT$ *in H and x is the label of some rule from* R_i.

We write $A\ {}_i\Rightarrow^* B$ *and say* B is achievable from A in i-th component of G *if and only if there exists derivation* $A \Rightarrow^* xB,\ x \in {}_HT^*$ *in H, and x are the labels of rules from* R_i.

We write $i(A) = \{B \mid B \in {}_HN$ *and* $A\ {}_i\Rightarrow^* B\}$.

THEOREM 4.9. *For any* $n, k \ge 1$, $\mathfrak{R}(REG, n, k) \subseteq REG$. *That is, let* $G = (N_1, N_2, \ldots, N_n, T, R_1, R_2, \ldots, R_n, S)$ *be an* $n-_{dis}RLG$ *regulated by regular language* Ξ. *Let generation of* $L(G, \Xi)$ *by G require no more than k changes of derivation positions. Then, there exists an equivalent regular grammar* $G' = (N', T', S', P')$ *such that* $L(G, \Xi) = L(G')$.

PROOF. *Let* $\Xi = L(H), H = ({}_HN, {}_HT, {}_HS, {}_HP)$, $N = N_1 \cup N_2 \cup \ldots \cup N_n$, *and* $S = S_1S_2\ldots S_n$.
We will construct set $\widehat{N}$ *in this way:*

- *if* ${}_HS\ {}_i\Rightarrow^* A$ *in H,* $A \in {}_HN$, *add* $\langle \varepsilon, \varepsilon, \ldots, {}_HSA\#, \ldots, \varepsilon \rangle$ *to* $\widehat{N}$, *where* ${}_HSA\#$ *is at the ith position.*

- *if* $C\ {}_j\Rightarrow^* A$ *and* $A\ {}_i\Rightarrow^* B$ *in H,* $i < j$, $A, B, C \in {}_HN$ *and* $\langle y_1, y_2, \ldots, y_i, \ldots, y_j, \ldots, y_n \rangle \in \widehat{N}$, *such that* $y_j \in \{{}_HN_HN\}^*\{C\}\{A\}\{\#\}$, *then add* $\langle y_1, y_2, \ldots, y_iAB\#, \ldots, y_j, \ldots, y_n \rangle$ *to* $\widehat{N}$.

- *if* $C\ {}_j\Rightarrow^* A$ *and* $A\ {}_i\Rightarrow^* B$ *in H,* $i > j$, $A, B, C \in {}_HN$ *and* $\langle y_1, y_2, \ldots, y_j, \ldots, y_i, \ldots, y_n \rangle \in \widehat{N}$, *such that* $y_j \in \{{}_HN_HN\}^*\{C\}\{A\}\{\#\}$, *then add* $\langle y_1, y_2, \ldots, y_j, \ldots, y_iAB\#, \ldots, y_n \rangle$ *to* $\widehat{N}$.

- *if* $A\ {}_i\Rightarrow x$ *in H,* $A \in {}_HN, x \in {}_HT$ *and* $\langle y_1, y_2, \ldots, y_i, \ldots, y_n \rangle \in \widehat{N}$, *such that* $y_i \in \{{}_HN_HN\}^*{}_HN\{A\}\{\#\}$, *then add* $\langle y_1, y_2, \ldots, y_i\bullet, \ldots, y_n \rangle$ *to* $\widehat{N}$.

The construction of $\widehat{N}$ *is completed.*
$\widehat{M} = \{X \mid X \in \widehat{N}, sub(X) \cap \{\bullet\} \ne \emptyset\}$.

Next, we construct grammar $G' = (N', T', S', P')$ *as follows:*

1. *if* $X \in \widehat{M}$*, then add* $[X, S_1]$ *to* N' *and* $S' \to [X, S_1]$ *to* P'.

2. *if*

$$(*)\quad \begin{array}{c} X = \langle y_1, y_2, \ldots, y_i, \ldots, y_n\rangle \in \widehat{M}, y_h = \varepsilon, \\ 0 \le h \le i-1, y_i = AB\#\overline{y}_i,\ A, B \in {}_HN \text{ and} \\ A\ {}_i\Rightarrow^* C\ {}_i\Rightarrow D\ {}_i\Rightarrow^+ B \text{ and } C \Rightarrow qD \text{ in } H \text{ and} \\ Y \Rightarrow aZ\ [q] \text{ in } G, \end{array}$$

   then add $[X, Z]$ *to* N' *and rule* $[X, Y] \to a[X, Z]$ *to* P'.

3. *if* $(*)$ *is untrue and if* $X = \langle y_1, y_2, \ldots, y_i, \ldots, y_n\rangle \in \widehat{M}, y_h = \varepsilon, 0 \le h \le i-1, y_i = AB\#\overline{y}_i,\ A, B \in {}_HN$ *and* $A\ {}_i\Rightarrow^* C\ {}_i\Rightarrow B$ *and* $C \Rightarrow qB$ *in* H *and* $Y \Rightarrow aZ\ [q]$ *in* G*, then add* $[\langle y_1, y_2, \ldots, \overline{y}_i, \ldots, y_n\rangle, Z]$ *to* N' *and rule* $[X, Y] \to a[\langle y_1, y_2, \ldots, \overline{y}_i, \ldots, y_n\rangle, Z]$ *to* P' *and replace* $\langle y_1, y_2, \ldots, y_i, \ldots, y_n\rangle$ *with* $\langle y_1, y_2, \ldots, \overline{y}_i, \ldots, y_n\rangle$ *in* $\widehat{M}$.

4. *if* $(*)$ *is untrue and if* $X = \langle y_1, y_2, \ldots, y_i, \ldots, y_n\rangle \in \widehat{M}, y_h = \varepsilon, 0 \le h \le i-1, y_i = AB\#,\ A, B \in {}_HN$ *and* $A\ {}_i\Rightarrow^* C\ {}_i\Rightarrow B$ *and* $C \Rightarrow qB$ *in* H *and* $Y \Rightarrow a\ [q]$ *in* G*, then add* $[\langle y_1, y_2, \ldots, \varepsilon, \ldots, y_n\rangle, S_{i+1}]$ *to* N' *and rule* $[X, Y] \to a[\langle y_1, y_2, \ldots, \varepsilon, \ldots, y_n\rangle, S_{i+1}]$ *to* P' *and replace* $\langle y_1, y_2, \ldots, y_i, \ldots, y_n\rangle$ *with* $\langle y_1, y_2, \ldots, \varepsilon, \ldots, y_n\rangle$ *in* $\widehat{M}$.

   Suppose that $S_{n+1} = \varepsilon$.

5. *Add* $[\langle\varepsilon, \ldots, \varepsilon\rangle, \varepsilon] \to \varepsilon$ *to* P'.

6. *If* $X = [\langle\varepsilon, \ldots, \varepsilon, \bullet, y_i, \ldots, y_n\rangle, Y] \in \widehat{M}$*, then replace* X *with* $[\langle\varepsilon, \ldots, \varepsilon, \varepsilon, y_i, \ldots, y_n\rangle, Y]$ *in* $\widehat{M}$.

7. $T' = T$.

Next we prove that $L(G, \Xi, k) = L(G')$.
$L(G, \Xi, k) \subseteq L(G')$ *: for every* $w \in L(G, \Xi, k)$*, there exists a derivation of the form*

$$(1)\quad \begin{array}{l} {}_HS\ {}_{i_1}\Rightarrow^* q_1A_1\ {}_{i_2}\Rightarrow^* q_1q_2A_2\ {}_{i_3}\Rightarrow^* \ldots {}_{i_{p-1}}\Rightarrow^* \\ {}_{i_{p-1}}\Rightarrow^* q_1 \ldots q_{p-1}A_{p-1}\ {}_{i_p}\Rightarrow^* q_1 \ldots q_p = q \text{ in } H \end{array}$$

and

$$\begin{array}{l} S_1 \ldots S_{i_1} \ldots S_{i_2} \ldots S_n \Rightarrow^* S_1 \ldots w_1X_1 \ldots S_{i_2} \ldots S_n\ [q_1] \Rightarrow^* \\ \quad \Rightarrow^* S_1 \ldots w_1X_1 \ldots w_2X_2 \ldots S_n\ [q_2] \Rightarrow^* \ldots \Rightarrow^* \\ \quad \Rightarrow^* w_{i_1} \ldots w_{i_{p-1}} \ldots \overline{w}_{i_p}X_{p-1} \ldots w_{i_n}\ [q_{p-1}] \Rightarrow^* \\ \quad \Rightarrow^* w_{i_1} \ldots w_{i_n}\ [q_p] = w \end{array}$$

in G*, where* $q_h \in R_h, 1 \le h \le n$.
Derivation (1) *can be rewritten in this form*

$$X = \langle y_1, y_2, \ldots, {}_HSA_1\#\overline{y}_{i_1}, \ldots, A_1A_2\#\overline{y}_{i_2}, \ldots, \overline{y}_p\#\bullet, \ldots, y_n\rangle$$

which belongs to $\widehat{M}$*. We start derivation in* G' *from start symbol* $[X, S_1]$.

$$\begin{array}{l} [X, S_1] = [\langle y_1, y_2, \ldots, {}_HSA_1\#\overline{y}_{i_1}, \ldots \\ \qquad \ldots A_1A_2\#\overline{y}_{i_2}, \ldots, \overline{y}_p\#\bullet, \ldots, y_n\rangle, S_1] \Rightarrow^* \\ \quad \Rightarrow^* w_{i_1}[\langle\varepsilon, y_2, \ldots, {}_HSA_1\#\overline{y}_{i_1}, \ldots \\ \qquad \ldots A_1A_2\#\overline{y}_{i_2}, \ldots, \overline{y}_p\#\bullet, \ldots, y_n\rangle, S_2] \Rightarrow^* \\ \quad \Rightarrow^* \ldots \Rightarrow^* w_{i_1} \ldots w_{i_{n-1}}[\langle\varepsilon, \ldots, \varepsilon, y_n\rangle, S_n] \Rightarrow^* \\ \quad \Rightarrow^* w_{i_1} \ldots w_{i_n}[\langle\varepsilon, \ldots, \varepsilon\rangle, \varepsilon] \Rightarrow w_{i_1} \ldots w_{i_n} = w. \end{array}$$

Hence, a $w \in L(G, \Xi, k)$ *implies* $w \in L(G')$.
$L(G, \Xi, k) \supseteq L(G')$ *: for every* $w \in L(G')$*, there exists a successful derivation from start symbol* $[X, S_1]$

$$\begin{array}{l} [X, S_1] = [\langle y_1, y_2, \ldots, {}_HSA_1\#\overline{y}_{i_1}, \ldots \\ \qquad \ldots A_1A_2\#\overline{y}_{i_2}, \ldots, \overline{y}_p\#\bullet, \ldots, y_n\rangle, S_1] \Rightarrow^* \\ \quad \Rightarrow^* w_{i_1}[\langle\varepsilon, y_2, \ldots, {}_HSA_1\#\overline{y}_{i_1}, \ldots \\ \qquad \ldots A_1A_2\#\overline{y}_{i_2}, \ldots, \overline{y}_p\#\bullet, \ldots, y_n\rangle, S_2] \Rightarrow^* \\ \quad \Rightarrow^* \ldots \Rightarrow^* w_{i_1} \ldots w_{i_{n-1}}[\langle\varepsilon, \ldots, \varepsilon, y_n\rangle, S_n] \Rightarrow^* \\ \quad \Rightarrow^* w_{i_1} \ldots w_{i_n}[\langle\varepsilon, \ldots, \varepsilon\rangle, \varepsilon] \Rightarrow w_{i_1} \ldots w_{i_n} = w. \end{array}$$

$X \in \widehat{M}$ *is of the form*

$$X = \langle y_1, y_2, \ldots, {}_HSA_1\#\overline{y}_{i_1}, \ldots, A_1A_2\#\overline{y}_{i_2}, \ldots, \overline{y}_p\#\bullet, \ldots, y_n\rangle$$

X *defines the derivation*

$$\begin{array}{l} {}_HS\ {}_{i_1}\Rightarrow^* q_1A_1\ {}_{i_2}\Rightarrow^* q_1q_2A_2\ {}_{i_3}\Rightarrow^* \ldots {}_{i_{p-1}}\Rightarrow^* \\ {}_{i_{p-1}}\Rightarrow^* q_1 \ldots q_{p-1}A_{p-1}\ {}_{i_p}\Rightarrow^* q_1 \ldots q_p = q \text{ in } H \end{array}$$

which regulates grammar G *in this way*

$$\begin{array}{l} S_1 \ldots S_{i_1} \ldots S_{i_2} \ldots S_n \Rightarrow^* S_1 \ldots w_1X_1 \ldots S_{i_2} \ldots S_n\ [q_1] \Rightarrow^* \\ \quad \Rightarrow^* S_1 \ldots w_1X_1 \ldots w_2X_2 \ldots S_n\ [q_2] \Rightarrow^* \ldots \Rightarrow^* \\ \quad \Rightarrow^* w_{i_1} \ldots w_{i_{p-1}} \ldots \overline{w}_{i_p}X_{p-1} \ldots w_{i_n}\ [q_{p-1}] \Rightarrow^* \\ \quad \Rightarrow^* w_{i_1} \ldots w_{i_n}\ [q_p] = w \end{array}$$

in G*, where* $q_h \in R_h, 1 \le h \le n$.
Thus, $w \in L(G, \Xi, k)$ *so* $L(G, \Xi, k) = L(G')$.

Because $card(N') < card({}_HN)^{2^{(n*k)}}$ *and all rules are regular,* $G' \in REG$. □

As opposed to Theorem 4.6, the next theorem demonstrates that if during the derivation of any sentence from the generated language, these grammars change the position of rewriting finitely many times, then they always generate only the family of regular languages independently of the length of their start strings.

THEOREM 4.10.

$$\mathfrak{R}(REG, n, k) = REG.$$

PROOF. $REG = \mathfrak{R}(REG, 1, 0) \subseteq \mathfrak{R}(REG, n, k) \subseteq REG$ *(see Theorem 4.9).* □

5. REFERENCES

[1] S. Arto. *Formal Languages.* Academic Press, 1973.

[2] J. Dassow and G. Paun. *Regulated rewriting in formal language theory*, volume 18 of *EATCS Monographs.* Springer-Verlag, 1989.

[3] S. A. Greibach. An infinite hierarchy of context-free languages. *Journal of the ACM*, 16(1):91–106, Jan. 1969.

[4] L. Y. Liu and P. Weiner. An infinite hierarchy of intersections of context-free languages. *mst*, 7(2):185–192, 1972.

[5] A. Meduna. *Automata and Languages: Theory and Applications.* Springer, London, 2005.

[6] G. Rozenberg and A. Salomaa, editors. *Handbook of Formal Languages (3 volumes).* Springer, 1997.

[7] D. Wood. Properties of n-parallel finite state languages. Technical Report 73/4, McMaster University, Hamilton, Canada, 1973.