On the Network Properties of Generalized Hypercubes

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ABSTRACT
A generalized hypercube (GHC) interconnection network is based on a mixed radix number system and supports any number of processors. For a given number of processors, it is possible to design a GHC multiprocessor sysem in several ways. From these, one can choose the best design to achieve the performance goals with the available cost. Some of its powerful interconnection properties are: low diameter, high connectivity, high fault tolerance and rich embeddings. Because of these features, it forms the basis of an ideal parallel architecture. Thus, the analysis of their topological properties is important. In this study, we focus on three of these properties: the bisection width, cut width and total edge length. Finally, we obtain their exact values for any generalized hypercube.

1. INTRODUCTION
Due to advances in technology, it is now feasible to build large-scale parallel computers. A network is an essential component in a large-scale parallel computer because it provides communication among the processors and/or the memories. Since network topology significantly affects system performance, one crucial step on designing a large-scale parallel computer is to determine the topology of the interconnection network.

The hypercube has been studied extensively as a network topology for parallel computers. Many important problems such as sorting, prefix computation, fast Fourier transform, matrix multiplication, matrix transposition, matrix inversion, eigenvalues, connected components, all-pairs shortest paths, minimum spanning trees, and more, can be efficiently solved in the hypercube. Aside from that, many other networks such as linear arrays, meshes and trees, among others, can be effectively embedded in the hypercube.

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The generalized hypercube structure was introduced by Bhuyan and Agrawal in [4]. It can support any number of processors instead of just a power of 2. For a given number of processors, it is possible to design a GHC multiprocessor structure in several ways. The design is based on the allowable diameter of the network. If the diameter can be increased, a structure with a lower degree can be obtained. These structures are highly fault-tolerant, they possess a small average message distance and a flow traffic density.

This paper is organized as follows: In Section 2, we provide some definitions and notations essential to the paper. In Section 3 we state some lemmas and theorems needed in the proof of the main results. Finally, we prove our main results in Section 4.

2. PRELIMINARIES
In building an interconnection network, it is important to have a good network design. We need to consider how processors are interconnected to create a system, ideally, with low degree, regularity, small diameter, large bisection width and high fault tolerance. That is why it is significant to study network properties. In this section, we define some of these properties and additional definitions that are necessary in our discussion. Most of these are standard in computer science ([3],[8],[13]).

Definition 1. *$G = (V, E)$ represents an undirected graph where V is the set of nodes or processors and E is the set of edges or communication links. An unordered pair (u, v) represents an undirected edge in E connecting nodes u and v in V. The total number of nodes in G is $|V|$ and the total number of links in G is $|E|$.*

Definition 2. *Two nodes $u, v \in V$ are connected if there exists a path that joins them. That is, there is a series of nodes $u_1, u_2, u_3, \cdots, u_k \in V$ such that (u, u_1), (u_1, u_2), (u_2, u_3), $\cdots$, (u_k, v) are edges in E. The distance between u and v is the length of a path from u to v that has the minimum length among all such paths.*

Definition 3. *The diameter of a graph G is the maximum distance between any two of its nodes.*

Definition 4. *Given a graph G, the degree of a node $v \in$*

V is the number of edges incident with v. A graph G is r-regular if every node has degree r.

Note that the diameter is an upper bound on the number of communication steps needed to relay information between any two processors. Thus, in an interconnection network, we clearly prefer a small diameter. We also prefer a small maximum degree because it means that the interconnection network is easier to build.

DEFINITION 5. *The node-connectivity of a graph G is the minimum number of nodes in V whose removal results in a disconnected or trivial graph.*

DEFINITION 6. *A subgraph H of G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Given a nonempty subset S of $V(G)$, the subgraph $G[S]$ induced by S is the subgraph of G whose node set is S and whose edge set is the set of all edges of G that have both end-nodes in S.*

DEFINITION 7. *The fault diameter of a graph G of node-connectivity c is the maximum diameter over all subgraphs of G obtained by removing less than c nodes. The fault diameter is frequently used to measure the fault tolerance of an interconnection network.*

DEFINITION 8. *A linear layout of an undirected graph $G = (V, E)$ with $n = |V|$ vertices is a bijective function $l : V \rightarrow \{0, \cdots, n-1\}$. The natural order layout is a linear layout N such that $N(u) = u$ for every $u \in \{0, \cdots, n-1\}$. Note that a linear layout can take any of the $n!$ permutations, not just the natural order layout.*

DEFINITION 9. *The width of a graph G under a linear layout l at a gap i denoted by $C(G, l, i)$ is a set of edges connecting a node at a position less than i and one at a position larger than or equal to i. That is,*

$$C(G, l, i) = \{(u, v) \in E | 0 \leq l(u) < i \leq l(v) \leq |V| - 1\}.$$

Furthermore, we define $C^-(G, l, i)$ (resp. $C^+(G, l, i)$) as the set of edges connecting nodes whose positions are less than (resp. larger than or equal to) i. That is,

$$C^-(G, l, i) = \{(u, v) \in E | 0 \leq l(u) \leq l(v) < i\}.$$

$$C^+(G, l, i) = \{(u, v) \in E | i \leq l(u) \leq l(v) \leq |V| - 1\}.$$

Thus, the set E of edges in G is partitioned into three sets $C(G, l, i)$, $C^-(G, l, i)$ and $C^+(G, l, i)$. Hence, $|E| = |C(G, l, i)| + |C^-(G, l, i)| + |C^+(G, l, i)|$.

DEFINITION 10. *The bisection width $BW(G)$ of a graph G is the minimum number of edges which must be removed to separate the graph into two disjoint and equal-sized subgraphs. In terms of the width of a graph under a linear layout, $BW(G)$ is the minimum number of edges in $C(G, l, \lfloor |V| \rfloor / 2)$ over all linear layouts. That is,*

$$BW(G) = \min_l |C(G, l, \lfloor |V| \rfloor / 2)|.$$

DEFINITION 11. *The cut width of a graph G under a linear layout l is the maximum of $|C(G, l, i)|$ over all gaps i. The cut width $CW(G)$ of a graph G is the minimum cut width over all linear layouts. That is,*

$$CW(G) = \min_l \max_i |C(G, l, i)|.$$

DEFINITION 12. *The length of edge $(u, v) \in E$ under a linear layout l is $|l(u) - l(v)|$. The total edge length of a graph G under a linear layout l is $\sum_{(u,v) \in E} |l(u) - l(v)|$. Furthermore, the total edge length $TL(G)$ of a graph G is defined as the minimum total edge length over all linear layouts. That is,*

$$TL(G) = \min_l \sum_{(u,v) \in E} |l(u) - l(v)|.$$

Before we can define the network structure that we are interested in, i.e. the generalized hypercube, we first describe how the nodes of the structure are labelled.

DEFINITION 13. *(A Mixed Radix Number System) Let n be represented as a product of m_i's, $m_i > 1$ for $1 \leq i \leq d$. That is, $n = m_d * m_{r-1} * \cdots * m_1$. Then each number from 0 to $n-1$ can be expressed as a d-tuple $(x_d x_{d-1} \cdots x_1)$ for $0 \leq x_i \leq m_i - 1$. Associated with each x_i is a weight w_i such that $\sum_{i=1}^{d} x_i w_i = X$ where $w_1 = 1$ and $w_i = \prod_{j=1}^{i-1} m_j = m_{i-1} * m_{i-2} * \cdots * m_1$ for all $2 \leq i \leq d$.*

EXAMPLE 1. *Let $n = 16 = 4 * 2 * 2$.*

$$m_1 = 2 \quad m_2 = 2 \quad m_3 = 4$$
$$w_1 = 1 \quad w_2 = 2 \quad w_3 = 4$$

Then $X = (x_3 x_2 x_1)$, $0 \leq x_1 \leq 1$, $0 \leq x_2 \leq 1$, $0 \leq x_3 \leq 3$, for $X = 0, 1, 2, \cdots, 15$. Hence, $0 = (000)$, $9 = (201)$ and $15 = (311)$ in this mixed radix system.

DEFINITION 14. *(The Generalized Hypercube, GHC) Let n be the total number of processors. In a generalized hypercube structure, each processor X is represented by a d-tuple in the mixed radix number system. The processor $X = (x_d x_{d-1} \cdots x_{i+1} x_i x_{i-1} \cdots x_1)$ is connected to processors $X' = (x_d x_{d-1} \cdots x_{i+1} x_i' x_{i-1} \cdots x_1)$ for all $1 \leq i \leq d$ where $x_i' \neq x_i$, $0 \leq x_i' \leq m_i - 1$.*

Hence, the GHC structure consists of d-dimensions with m_i number of nodes in the i^{th} dimension. A node in a particular axis is connected to all other nodes in the same axis. The distance between any two nodes is the number of coordinates in which the addresses differ. Since the addresses can differ in at most all the d coordinates, the diameter of the structure is d. Figure 1 shows the GHC structure of the example in section 13.

In this example, 16 is factored as $4*2*2$. This GHC structure can also be described as a product of cliques: $K_4 \times K_2 \times K_2$. Of course, it is also possible to express 16 as $8 * 2$, $4 * 4$ or $2 * 2 * 2 * 2$. These will produce new GHC structures with diameter 2, 2, and 4 respectively. In general, we say

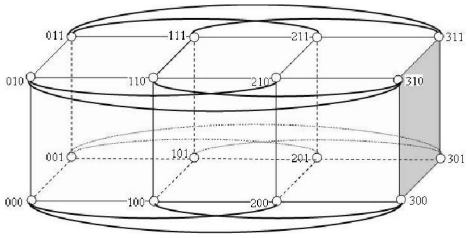

Figure 1: A three-dimensional generalized hypercube with 16 nodes.

that the d-dimensional GHC structure with m_i number of nodes in the i^{th} dimension is a product of d cliques: $K_{m_1} \times K_{m_2} \times \cdots K_{m_d}$. Hereon, we represent a d-dimensional GHC structure by $C^d = K_{m_1} \times K_{m_2} \times \cdots K_{m_d}$.

Network Properties of a GHC

A d-dimensional generalized hypercube $C^d = K_{m_1} \times K_{m_2} \times \cdots K_{m_d}$ has the following basic properties:

diameter	d	
no. of nodes	$\|V\|$	where $\|V\| = m_1 \cdot m_2 \cdots m_d$
regularity	r	where $r = \sum_{i=1}^{d} (m_i - 1)$
node-connectivity	r	
fault diameter	$d+1$	
no. of links	$\|E\|$	where $\|E\| = r(\|V\|/2)$

Observation 1. *A generalized hypercube has many symmetries. For any pair (u, v) and (u', v') there is an automorphism σ of G such that $\sigma(u) = u'$ and $\sigma(v) = v'$. There are many such automorphisms.*

Observation 2. *C^d is a Cayley graph and hence is vertex transitive.*

Definition 15. *A d-dimensional GHC structure with c nodes in each of the d dimensions is a product of d c-node cliques. It is also known as the d-dimensional c-ary clique C_c^d.*

In [10], Nakano obtained the exact values for the bisection width, cut width and total edge length of C_c^d and are stated in the following theorems.

Theorem 1. *The bisection width $BW(C_c^d)$ of C_c^d is*

$$\begin{cases} c^{d+1}/4 & if\ c\ is\ even \\ (c+1)(c^d-1)/4 & if\ c\ is\ odd. \end{cases}$$

Theorem 2. *The cut width $CW(C_c^d)$ of C_c^d is*

$$\begin{cases} c(c+2)(c^d-1)/4(c+1) & if\ c\ is\ even\ and\ d\ is\ even \\ c^2(c+2)(c^{d-1}-1)/4(c+1) & if\ c\ is\ even\ and\ d\ is\ odd \\ (c+1)(c^d-1)/4 & if\ c\ is\ odd. \end{cases}$$

Theorem 3. *The total edge length $TL(C_c^d)$ of C_c^d is* $TL(C_c^d) = (c+1)c^d(c^d-1)/6$

To the best of our knowledge, no exact values for the general case have been published as of the writing of this paper. Thus, in section 4, we obtain these values for any type of generalized hypercube.

3. FOR GENERALIZED HYPERCUBES

In this section, we restate some of the lemmas and theorems cited in [10]. Their proofs are similar to that in the paper mentioned. We will need the following for the proof of the main results.

Definition 16. *A subgraph of $C^d = K_{m_1} \times K_{m_2} \times \cdots \times K_{m_d}$ denoted by $C^{(n)}_{m_1 \times m_2 \times \cdots \times m_d}$ has n nodes labeled by n integers from 0 to $n-1$ which are connected in the same way as C^d. $C^{(n)}_{m_1 \times m_2 \times \cdots \times m_d}$ is an induced subgraph of $C^{(m)}_{m_1 \times m_2 \times \cdots \times m_d}$ whenever $n \le m$.*

Lemma 1. *Let f_j be the function defined as follows:*

$$f_j(n) = \begin{cases} n(n-1)/2 & if\ n \le m_j \\ \sum_{i=0}^{m_j-1} \left\{ f_{j+1}\left(\left\lfloor \frac{n+i}{m_j} \right\rfloor \right) + (m_j - i - 1) \left\lfloor \frac{n+i}{m_j} \right\rfloor \right\} & if\ n > m_j \end{cases}$$

For all $n \ge 1$, $C^{(n)}_{m_1 \times m_2 \times \cdots \times m_d}$ has exactly $f_1(n)$ edges.

Lemma 2. *Let g_j be the function defined as follows:*

$$g_j(n) = \begin{cases} n(n-1)/2 & if\ n \le m_j \\ max \left\{ \sum_{i=0}^{m_j-1} \{ f_{j+1}(n_i) + (m_j - i - 1) n_i \} \mid n_0 \le n_1 \le \cdots \le n_{m_j-1} < n = \sum_{i=0}^{m_j-1} n_i \right\} & if\ n > m_j \end{cases}$$

For any subgraph $G = (V, E)$ of $C^{(m)}_{m_1 \times m_2 \times \cdots \times m_d}$, $|E| \le g_1(|V|)$ always holds.

Lemma 3. *For every n, $f_1(n) = g_1(n)$ always holds.*

Theorem 4. *$C^{(n)}_{m_1 \times m_2 \times \cdots \times m_d}$ is a maximum n-node subgraph of $C^{(m)}_{m_1 \times m_2 \times \cdots \times m_d}$ whenever $n \le m$.*

Proof. Follows from the preceeding three lemmas. □

Lemma 4. *For any linear layout l and gap i,*
$$|C(C^d, l, i)| \ge |C(C^d, N, i)|.$$

Proof. From Defn 9 and Defn 16, it is easy to show that $|C^-(C^d, N, i)| = |C^{(i)}_{m_1 \times m_2 \times \cdots \times m_d}|$. Since the natural order layout of C^d has bilateral symmetry, we also have $|C^+(C^d, N, i)| = |C^-(C^d, N, n-i)| = |C^{(n-i)}_{m_1 \times m_2 \times \cdots \times m_d}|$. Note that for any linear layout l, $C^+(C^d, l, i)$ and $C^-(C^d, l, n-i)$ are $(n-i)$ and i-node subgraphs of C^d, respectively. From Thm 4, $C^+(C^d, l, i) \le |C^{(n-i)}_{m_1 \times m_2 \times \cdots \times m_d}|$ and $C^-(C^d, l, i) \le |C^{(i)}_{m_1 \times m_2 \times \cdots \times m_d}|$. It follows that $|C^+(C^d, l, i)| \le |C^+(C^d, N, i)|$

and $|C^-(C^d,l,i)| \leq |C^-(C^d,N,i)|$. Recall that for any graph G and linear layout l of G with E the set of edges of G, $|E| = |C(G,l,i)| + |C^-(G,l,i)| + |C^+(G,l,i)|$. Hence, $|C(C^d,l,i)| \geq |C(C^d,N,i)|$ for any linear layout l of C^d. □

As a result of Lemma 4, we have simplified the computation for the following parameters for C^d:

- $BW(C^d) = |C(C^d, N, \lfloor |V|/2 \rfloor)|$,
- $CW(C^d) = \max_i |C(C^d,N,i)|$,
- $TL(C^d) = \sum_{i=1}^{|V|-1} |C(C^d,N,i)|$.

We also define several notations that will be referred to in the next section.

Definition 17. *An edge is called a dimension k edge if it links two nodes that differ in the k^{th} bit position. Particularly, we define $C^d[k]$ as the set of all dimension k edges of C^d. That is,*

$$C^d[k] = \{(u,v) \in E(C^d) | u_k \neq v_k\} \text{ for } 1 \leq k \leq d$$

where $(u_d u_{d-1} \cdots u_1)$ and $(v_d v_{d-1} \cdots v_1)$ are mixed radix representations of nodes u and v respectively. Hence, $E(C^d)$ is partitioned into d subsets: $C^d[1], C^d[2], \cdots, C^d[d]$. Furthermore, we let $C(C^d[k],l,i)$ denote a set of edges defined as follows:

$$C(C^d[k],l,i) = C(C^d,l,i) \cap C^d[k].$$

That is, $C(C^d[k],l,i)$ is the set of all dimension k edges which are separated at gap i under linear layout l.

4. MAIN RESULTS

We now extend the theorems stated in section 2 to any generalized hypercube.

Theorem 5. *Given a d-dimensional generalized hypercube C^d where $C^d = K_{m_1} \times K_{m_2} \times \cdots \times K_{m_d}$ with $m_1 \leq m_2 \leq \cdots \leq m_d$. Let e be the largest index for which m_e is even. Set $e = 1$ if each factor has odd size. Then,*

$$BW(C^d) = \sum_{i=e}^{d} BW(K_{m_i}) M_i,$$

where $M_i = m_{i-1} m_{i-2} \cdots m_1$ for $2 \leq i \leq d$ with $M_1 = 1$.

Note that

$$BW(K_{m_i}) = \begin{cases} m_i^2/4 & \text{if } m_i \text{ is even} \\ (m_i^2-1)/4 & \text{if } m_i \text{ is odd.} \end{cases}$$

Proof. C^d has m_d $C^{d-1} = K_{m_1} \times K_{m_2} \times \cdots \times K_{m_{d-1}}$. Furthermore, the corresponding nodes of the m_d C^{d-1}'s are connected by K_{m_d}. Thus the m_d C^{d-1}'s are connected by M_d K_{m_d}'s. Similarly, each of the C^{d-1} has m_{d-1} $C^{d-2} = K_{m_1} \times K_{m_2} \times \cdots \times K_{m_{d-2}}$ where the m_{d-1} C^{d-2}'s are connected by M_{d-1} $K_{m_{d-1}}$'s. We repeat this procedure until we reach the 1-dimensional $C^1 = K_{m_1}$ which has m_1 nodes where each pair of nodes are connected by an edge.

Suppose that m_d is even. Then to separate the graph into two disjoint and equal sized subgraphs we only need to group together $\frac{m_d}{2}$ C^{d-1} to form one subgraph and the rest will form the other subgraph. Since the m_d C^{d-1}'s of C^d are connected by M_d K_{m_d}'s, the problem of finding the bisection width of C^d is reduced to computing for the bisection width of each of the M_d K_{m_d}'s. That is, $BW(C^d) = M_d \cdot BW(K_{m_d})$.

Suppose that m_d is odd. We group together $\frac{m_d-1}{2}$ C^{d-1} and another $\frac{m_d-1}{2}$ C^{d-1} in a separate group. To divide them, we need to divide each of M_d K_{m_d}'s connecting them. But there is 1 remaining C^{d-1} that needs to be split into the two groups. Recall that C^{d-1} has m_{d-1} C^{d-2}'s which are connected by M_{d-1} $K_{m_{d-1}}$'s. If m_{d-1} is even, $\frac{m_{d-1}}{2}$ C^{d-2} will be in the first group and the rest will be in the second group. To divide them, we need to divide each of the M_{d-1} $K_{m_{d-1}}$'s connecting them. Hence, $BW(C^d) = M_d \cdot BW(K_{m_d}) + M_{d-1} \cdot BW(K_{m_{d-1}})$. But if m_{d-1} is odd, we do a similar procedure as above until we have either an even m_e or m_1.

Thus, $BW(C^d) = M_d \cdot BW(K_{m_d}) + M_{d-1} \cdot BW(K_{m_{d-1}}) + \cdots + M_e \cdot BW(K_{m_e})$ where e is the largest index for which m_e is even or $e = 1$ if all of the m_i's are odd. □

Theorem 6. *Given a d-dimensional generalized hypercube C^d where $C^d = K_{m_1} \times K_{m_2} \times \cdots \times K_{m_d}$ with $m_1 \leq m_2 \leq \cdots \leq m_d$. Then,*

$$CW(C^d) = \sum_{k=1}^{d} \left(M_k i_k (m_k - i_k) + \left(\sum_{j=1}^{k-1} M_j i_j \right) (m_k - 2i_k - 1) \right)$$

where $M_s = m_{s-1} m_{s-2} \cdots m_1$ for $2 \leq s \leq d$ with $M_1 = 1$ and

if at least one m_s is odd, then $i_k = \begin{cases} \frac{m_k-1}{2} & \text{if } m_k \text{ is odd} \\ \frac{m_k}{2} & \text{if } m_k \text{ is even.} \end{cases}$

else, $\begin{cases} i_{d-2q} = \frac{m_{d-2q}}{2} \\ i_{d-2q-1} = \frac{m_{d-2q}}{2} - 1 \end{cases}$ *for $q \in \{0, 1, \cdots, \frac{d}{2}+1\}$*

Proof.
Let $i_d i_{d-1} \cdots i_1$ be the mixed radix representation of i.
Note that for all $k \in 1 \cdots d$, $|C(C^d[k],N,i)|$
$= [m_1 \ldots m_{d-1} i_d + m_1 \ldots m_{d-2} i_{d-1} + \cdots + m_1 \ldots m_{k-1} i_k + m_1 \ldots m_{k-1} - i](m_k - i_k) i_k + [i - m_1 \ldots m_{k-1} i_k - \cdots - m_1 \ldots m_{d-1} i_d](m_k - i_k - 1)(i_k + 1)$

$$= \left(\sum_{j=k}^{d} M_j i_j + M_k - i \right) (m_k - i_k) i_k + \left(i - \sum_{j=k}^{d} M_j i_j \right) (m_k - i_k - 1)(i_k + 1)$$

$$= \left(M_k i_k (m_k - i_k) + \left(\sum_{j=1}^{k-1} M_j i_j \right) (m_k - 2i_k - 1) \right).$$

Hence, we have $|C(C^d[k],N,i+1)| - |C(C^d[k],N,i)| = m_k - 2i_k - 1$ for every k. That is, the width at gap i changes by $m_k - 2i_k - 1$ as i increases.
Suppose that m_k is odd. Then $|C(C^d[k],N,i)|$ is maximum when $m_k - 2i_k - 1 = 0$, i.e, $i_k = \frac{m_k-1}{2}$. Now let's suppose that m_k is even. To maximize $|C(C^d[k],N,i)|$, we select either $i_k = \frac{m_k}{2}$ or $i_k = \frac{m_k}{2} - 1$.
Next, we assume that for all k, m_k is even. Note that $|C(C^d[d],N,i+1) \cup C(C^d[d-1],N,i+1)| - |C(C^d[d],N,i) \cup$

$C(C^d[d-1], N, i)| = m_d - 2i_d - 1 + m_{d-1} - 2i_{d-1} - 1$ and is maximal when $m_d - 2i_d + m_{d-1} - 2i_{d-1} - 2 = 0$. Thus, we should select $i_d = \frac{m_d}{2}$ and $i_{d-1} = \frac{m_{d-1}}{2} - 1$. Similarly, we have $i_{d-2} = \frac{m_{d-2}}{2}$ and $i_{d-3} = \frac{m_{d-2}}{2} - 1$. And $i_{d-2q} = \frac{m_{d-2q}}{2}$, $i_{d-2q-1} = \frac{m_{d-2q}}{2} - 1$ for $q \in \{0, 1, \cdots, \frac{d}{2} + 1\}$. $\square$

THEOREM 7. *Given a d-dimensional generalized hypercube C^d where $C^d = K_{m_1} \times K_{m_2} \times \cdots \times K_{m_d}$ with $m_1 \le m_2 \le \cdots \le m_d$. Then,*
$TL(C^d) = \sum_{i=1}^{d} M_i \prod_{j \ne i} m_j TL(K_{m_i})$,
where $M_i = m_{i-1} m_{i-2} \cdots m_1$ for $2 \le i \le d$ with $M_1 = 1$.

Note that $TL(K_c) = \sum_{i=1}^{c-1} i(c-i)$.

PROOF. Since C^d has $\prod_{j \ne 1} m_j$ K_{m_1}'s, in the first dimension we have a total edge length of $\prod_{j \ne 1} m_j \cdot TL(K_{m_1})$.
In general, for all $i \in \{1, \cdots, d\}$, since C^d has $\prod_{j \ne i} m_j$ K_{m_i}'s, we have a total edge length of $M_i \cdot \prod_{j \ne i} m_j \cdot TL(K_{m_i})$. Note that adjacent nodes of C^d in the i^{th} dimension are of distance M_i in the natural order linear layout.
Summing up, we have $TL(C^d) = \sum_{i=1}^{d} M_i \prod_{j \ne i} m_j TL(K_{m_i})$. $\square$

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